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On Superconformal Characters and Partition Functions in Three Dimensions

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Possible short and semi-short positive energy, unitary representations of the $Osp(2N|4)$ superconformal group in three dimensions are discussed. Corresponding character formulae are obtained, consistent with character formulae for the $SO(3,2)$ conformal group, revealing long multiplet decomposition at unitarity bounds in a simple way. Limits, corresponding to reduction to various $Osp(2N|4)$ subalgebras, are taken in the characters that isolate contributions from fewer states, at a given unitarity threshold, leading to considerably simpler formulae. Via these limits, applied to partition functions, closed formulae for the generating functions for numbers of BPS operators in the free field limit of superconformal $U(n)\times U(n)$ $\mathcal{N}=6$ Chern Simons theory and its supergravity dual are obtained in the large n limit. Partial counting of semi-short operators is performed and consistency between operator counting for the free field and supergravity limits with long multiplet decomposition rules is explicitly demonstrated. Partition functions counting certain protected scalar primary semi-short operators, and their superconformal descendants, are proposed and computed for large n. Certain chiral ring partition functions are discussed from a combinatorial perspective.

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1. Introduction

With the resurgence of interest recently in three dimensional superconformal field theory, due largely to the discovery by Bagger and Lambert [1] of a new superconformal $\mathcal{N}=8$ Chern Simons theory and, more recently, by Aharony *et al* [2] of a superconformal $\mathcal{N}=6$ Chern-Simons theory, with $U(n)\times U(n)$ gauge symmetry, much attention has been devoted to uncovering dualities [3,2], investigating integrability [4,5] and spectra [6,7,8] *etc.* One issue that has hitherto perhaps not been explored in very much detail is the representation theory, underlying these theories, for positive energy, unitary representations of $Osp(\mathcal{N}|4)$. This paper is an attempt to address in some detail this issue and focuses on the case of even $\mathcal{N}$.[1]

For $\mathcal{N}=4$ superconformal symmetry in four dimensions such a detailed study of the representation theory [10] proved fruitful in many ways, *e.g.* investigating the operator product expansion, in terms of conformal partial wave expansions of four point functions, see [11], for example, in partially determining the operator spectrum of $\mathcal{N}=4$ super Yang Mills. In this context, superconformal characters provide for an alternative and arguably more straightforward and powerful way of organising the sometimes detailed multiplet structure that arises. They easily lead to the rules for long multiplet decomposition into short/semi-short multiplets at unitarity bounds, formulae for decompositions in terms of subalgebra representations and indicate how partition functions may be computed for decoupled sectors.

The outline of this paper is as follows. In section 2 the superconformal algebra for $Osp(2N|4)$, *i.e.* for $\mathcal{N}=2N$, is given in detail. Mainly for the purposes of discussing shortening conditions, constructing Verma module characters and finding subalgebras, it proves easiest to use the orthonormal basis for the $SO(2N)$ R-symmetry algebra. Section 3 gives a brief account of the possible shortening conditions for semi-short representations, denoted by (N,A,n), $n=1,\ldots,N{-}1$, and $(N,A,\pm)$, and short or BPS representations, denoted by (N,B,n) and $(N,B,\pm)$, as well as for the conserved current multiplet, denoted $(N,\mathrm{cons.})$. Here, highest weight states in the (N,A,n), respectively $(N,A,\pm)$, (N,B,n), $(N,B,\pm)$, representations are annihilated by $n/4N$, respectively $1/4$, $n/2N$, $1/2$ of the supercharges. (Hence, the $(N,B,\pm)$ representations are referred to here as half BPS. For $N=4$, it is well known that there are two types of half BPS representation, see [12], and

[1] Note that there is some overlap in the discussion here with some related issues explored in [9] particularly with regard to long multiplet decomposition formulae. Odd $\mathcal{N}$ would require some modification of the analysis here particularly with regard to possible short/semi-short multiplets. While straightforward, an extension to odd $\mathcal{N}$ is avoided here to ensure notation is not overly cumbersome.

[13] for a discussion.)

In section 4 the characters for positive energy, unitary representations of the $Osp(2N|4)$ superconformal group are obtained through use of Verma module characters and the Weyl symmetriser for the maximal compact subgroup, $U(1)\otimes SU(2)\otimes SO(2N)$. BPS and conserved conserved current multiplet characters are decomposed in terms of $SO(3,2)\otimes SO(2N)$ characters. The decomposition rules for long multiplets at the unitarity bounds are found. In particular, they imply that in any $\mathcal{N}=2N$ superconformal field theory in three dimensions, operators in all (N,B,n), $n>1$, as well as in $(N,B,\pm)$ and certain $(N,B,1)$, BPS multiplets must have protected conformal dimensions.

In section 5, various limits are taken in the characters that lead to non-vanishing formulae only for subsets of the characters corresponding to different short/semi-short multiplets. These are shown, from appendix A, to match with various $Osp(2N|4)$ subalgebra characters and, in terms of partition functions, isolate different sectors of operators.[2] For the subset being $\{(N,B,m),\ m\geq n,\ (N,B,+),(N,B,-)\}$ the subalgebra is $U(1)\otimes SO(2N{-}2m)$ while for the subsets $\{(N,B,\pm)\}$ there are two $U(1)$ subalgebras. Similarly, for the subset being $\{(N,A,m),\ m\geq n,\ (N,A,+),\ (N,A,-),\ (N,B,l),\ 1\leq l\leq N,\ (N,B,+),\ (N,B,-),\ (N,\text{cons.})\}$ where (N,B,l), $l<m$, multiplets have particular $SO(2N)$ R-symmetry eigenvalues, the subalgebra is $U(1)\otimes Osp(2N{-}2m|2)$. Finally, for the subsets $\{(N,A,\pm),\ (N,B,l),\ 1\leq l\leq N,\ (N,B,+),\ (N,B,-),\ (N,\text{cons.})\}$, where (N,B,l), $l<m$, $(N,B,\mp)$ multiplets have particular $SO(2N)$ R-symmetry eigenvalues, there are two $U(1)\otimes SU(1,1)$ subalgebras. The limits are shown to be consistent with long multiplet decomposition rules and also with the index of [6].

In section 6, the general free field multi-particle partition function for $\mathcal{N}=6$ superconformal Chern Simons theory is written in detail in terms of superconformal characters and, for large n, computed using symmetric polynomial techniques. Also, the corresponding partition function is given in detail for the supergravity limit.

In section 7, operator counting for short and certain semishort multiplets is investigated using the limits taken in section 5, and other symmetric polynomial techniques, described in appendix B, and shown to be consistent with expectations from long multiplet decomposition rules.

In the conclusion, partition functions counting certain protected semi-short operators are discussed. Some related comments are made about $\mathcal{N}=4$ super Yang Mills, explained in more detail in appendix C, where a certain class of chiral ring partition functions are

[2] I thank Troels Harmark for pointing out to me that different sectors, along with the $SU(2)\times SU(2)$ one, were also briefly considered in [5], for $\mathcal{N}=6$ superconformal Chern Simons theory.

computed using the Polya enumeration theorem.

2. The Superconformal Algebra in Three Dimensions

In d dimensions, the standard non-zero commutators of the conformal group $SO(d,2)$ are given by, for $\eta_{ab} = \text{diag.}(-1,1,\ldots,1)$, $a,b=0,1,\ldots,d-1$,

$$
\begin{aligned}
[M_{ab},P_c] &= i(\eta_{ac}P_b - \eta_{bc}P_a)\,, \qquad [M_{ab},K_c] = i(\eta_{ac}K_b - \eta_{bc}K_a)\,,\\
[M_{ab},M_{cd}] &= i(\eta_{ac}\,M_{bd} - \eta_{bc}\,M_{ad} - \eta_{ad}\,M_{bc} + \eta_{bd}\,M_{ac})\,,\\
[D,P_a] &= P_a\,, \quad [D,K_a] = -K_a\,, \quad [K_a,P_b] = -2iM_{ab} + 2\eta_{ab}D\,,
\end{aligned}
\tag{2.1}
$$

where the generators of translations are P_a, those of special conformal transformations are K_a, those of $SO(d-1,1)$ are $M_{ab} = -M_{ba}$ while that of scale transformations is D.

For later application it is convenient to write the $Sp(4,\mathbb{R})/\mathbb{Z}_2 \simeq SO(3,2)$ algebra for three dimensions in the spinor basis so that for,[3]

$$
P_{\alpha\beta} = (\gamma^a)_{\alpha\beta}\,P_a\,, \quad K^{\alpha\beta} = (\bar\gamma^a)^{\alpha\beta}\,K_a\,, \quad M_\alpha{}^\beta = \tfrac{i}{2}(\gamma^a\,\bar\gamma^b)_\alpha{}^\beta\,M_{ab}\,,
\tag{2.2}
$$

the algebra (2.1) becomes,

$$
\begin{aligned}
[M_\alpha{}^\beta, P_{\gamma\delta}] &= \delta_\gamma{}^\beta\,P_{\alpha\delta} + \delta_\delta{}^\beta\,P_{\alpha\gamma} - \delta_\alpha{}^\beta\,P_{\gamma\delta}\,,\\
[M_\alpha{}^\beta, K^{\gamma\delta}] &= -\delta_\alpha{}^\gamma\,K^{\beta\delta} - \delta_\alpha{}^\delta\,K^{\beta\gamma} + \delta_\alpha{}^\beta\,K^{\gamma\delta}\,,\\
[M_\alpha{}^\beta, M_\gamma{}^\delta] &= -\delta_\alpha{}^\delta\,M_\gamma{}^\beta + \delta_\gamma{}^\beta\,M_\alpha{}^\delta\,, \quad [D,P_{\alpha\beta}] = P_{\alpha\beta}\,, \quad [D,K^{\alpha\beta}] = -K^{\alpha\beta}\,,\\
[K^{\alpha\beta}, P_{\gamma\delta}] &= 4\,\delta_{(\gamma}{}^{(\alpha}\,M_{\delta)}{}^{\beta)} + 4\,\delta_{(\gamma}{}^{\alpha}\,\delta_{\delta)}{}^{\beta}\,D\,.
\end{aligned}
\tag{2.3}
$$

For supercharges and their superconformal extensions the non-zero (anti-)commutators are given by,

$$
\begin{aligned}
\{Q_{r\alpha},Q_{s\beta}\} &= 2\delta_{rs}\,P_{\alpha\beta}\,, & \{S_r{}^\alpha, S_s{}^\beta\} &= 2\delta_{rs}\,K^{\alpha\beta}\,,\\
[K^{\alpha\beta},Q_{r\gamma}] &= i(\delta_\gamma{}^\alpha\,S_r{}^\beta + \delta_\gamma{}^\beta\,S_r{}^\alpha)\,, & [P_{\alpha\beta},S_r{}^\gamma] &= -i(\delta_\alpha{}^\gamma\,Q_{r\beta} + \delta_\beta{}^\gamma\,Q_{r\alpha})\,,\\
[M_\alpha{}^\beta,Q_{r\gamma}] &= \delta_\gamma{}^\beta\,Q_{r\alpha} - \tfrac12\delta_\alpha{}^\beta\,Q_{r\gamma}\,, & [M_\alpha{}^\beta,S_r{}^\gamma] &= -\delta_\alpha{}^\gamma\,S_r{}^\beta + \tfrac12\delta_\alpha{}^\beta S_r{}^\gamma\,,\\
[D,Q_{r\alpha}] &= \tfrac12 Q_{r\alpha}\,, & [D,S_r{}^\alpha] &= -\tfrac12 S_r{}^\alpha\,,\\
[R_{rs},Q_{t\alpha}] &= i(\delta_{rt}\,Q_{s\alpha} - \delta_{st}\,Q_{r\alpha})\,, & [R_{rs},S_t{}^\alpha] &= i(\delta_{rt}\,S_s{}^\alpha - \delta_{st}\,S_r{}^\alpha)\,,
\end{aligned}
\tag{2.4}
$$

[3] The conventions used here for the gamma matrices are that $\gamma^0 = 1$, $\gamma^1 = \sigma_1$, $\gamma^2 = \sigma_3$, in terms of Pauli matrices σ_1,σ_3, so that γ^a are real, symmetric matrices. We take $(\bar\gamma^a)^{\alpha\beta} = \varepsilon^{\alpha\gamma}\,(\gamma^a)_{\gamma\delta}\,\varepsilon^{\delta\beta}$ where $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$, $\varepsilon^{12} = 1$, $\varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta} = -\delta_\alpha{}^\gamma\,\delta_\beta{}^\delta + \delta_\alpha{}^\delta\,\delta_\beta{}^\gamma$. We thus have $\gamma^a\bar\gamma^b + \gamma^b\bar\gamma^a = \eta^{ab}1$, $\bar\gamma^a\gamma^b + \bar\gamma^b\gamma^a = \eta^{ab}1$, and the completeness relation $(\gamma^a)_{\alpha\beta}(\bar\gamma_a)^{\gamma\delta} = \delta_\alpha{}^\gamma\,\delta_\beta{}^\delta + \delta_\alpha{}^\delta\,\delta_\beta{}^\gamma$.

along with

$$\{Q_{r\alpha}, S_s{}^\beta\} = 2i(M_\alpha{}^\beta \delta_{rs} - i\delta_\alpha{}^\beta\, R_{rs} + \delta_\alpha{}^\beta\, \delta_{rs}\, D)\,, \tag{2.5}$$

where the $SO(n)$ R-symmetry generators $R_{rs} = -R_{rs} = R_{rs}{}^\dagger$ satisfy

$$[R_{rs}, R_{tu}] = i(\delta_{rt}\, R_{su} - \delta_{st}\, R_{ru} - \delta_{ru}\, R_{st} + \delta_{su}\, R_{rt})\,. \tag{2.6}$$

In order to discuss highest weight states and shortening conditions (2.6) is now rewritten in terms of generators in the orthonormal basis of $SO(2N)$, which has rank N. In this basis, the Cartan subalgebra H_n, and raising/lowering operators $E^{+\pm}_{mn}/E^{-\pm}_{mn}$, $m < n$, are given by, for $m, n = 1, \dots, N$,[4]

$$\begin{aligned} H_n &= R_{2n-1\,2n}\,, \quad E^{+\pm}_{mn} = R_{2m-1\,2n-1} + i\, R_{2m\,2n-1} \pm i\, R_{2m-1\,2n} \mp R_{2m\,2n}\,, \\ E^{-\pm}_{mn} &= E^{+\mp\dagger}_{mn}\,, \qquad E^{-\pm}_{mn} = -E^{\pm -}_{nm}\,, \qquad E^{+\pm}_{mn} = -E^{\pm +}_{nm}\,, \end{aligned} \tag{2.7}$$

so that the non-zero commutators from (2.6) are, for $l = 1, \dots, N$,

$$\begin{aligned} [H_l, E^{+\pm}_{mn}] &= (\delta_{lm} \pm \delta_{ln}) E^{+\pm}_{mn}\,, \qquad [H_l, E^{-\pm}_{mn}] = (-\delta_{lm} \pm \delta_{ln}) E^{-\pm}_{mn}\,, \\ [E^{+\pm}_{mn}, E^{-\mp}_{mn}] &= 4(H_m \pm H_n)\,, \\ [E^{+\pm}_{lm}, E^{-\pm}_{ln}] &= 2i\, E^{\pm\pm}_{mn}\,, \qquad [E^{+\pm}_{lm}, E^{-\mp}_{ln}] = 2i\, E^{\pm\mp}_{mn}\,, \end{aligned} \tag{2.8}$$

where $m \neq n$ in the last line.

In this basis a convenient choice for the supercharges and their superconformal extensions is given by

$$\begin{aligned} \mathcal{Q}_{n\alpha} &= \frac{1}{\sqrt{2}}\Big(Q_{2n-1\,\alpha} + i\, Q_{2n\,\alpha}\Big)\,, \qquad \bar{\mathcal{Q}}_{n\alpha} = \frac{1}{\sqrt{2}}\Big(Q_{2n-1\,\alpha} - i\, Q_{2n\,\alpha}\Big)\,, \\ \mathcal{S}_n{}^\alpha &= \frac{1}{\sqrt{2}}\Big(S_{2n-1}{}^\alpha + i\, S_{2n}{}^\alpha\Big)\,, \qquad \bar{\mathcal{S}}_n{}^\alpha = \frac{1}{\sqrt{2}}\Big(S_{2n-1}{}^\alpha - i\, S_{2n}{}^\alpha\Big)\,. \end{aligned} \tag{2.9}$$

Trivially, from (2.4),

$$\begin{aligned} [K^{\alpha\beta}, \mathcal{Q}_{n\gamma}] &= i(\delta_\gamma{}^\alpha\, \mathcal{S}_n{}^\beta + \delta_\gamma{}^\beta\, \mathcal{S}_n{}^\alpha)\,, \qquad & [P_{\alpha\beta}, \mathcal{S}_n{}^\gamma] &= -i(\delta_\alpha{}^\gamma\, \mathcal{Q}_{n\beta} + \delta_\beta{}^\gamma\, \mathcal{Q}_{n\alpha})\,, \\ [M_\alpha{}^\beta, \mathcal{Q}_{n\gamma}] &= \delta_\gamma{}^\beta\, \mathcal{Q}_{n\alpha} - \tfrac{1}{2}\delta_\alpha{}^\beta\, \mathcal{Q}_{n\gamma}\,, \qquad & [M_\alpha{}^\beta, \mathcal{S}_n{}^\gamma] &= -\delta_\alpha{}^\gamma\, \mathcal{S}_n{}^\beta + \tfrac{1}{2}\delta_\alpha{}^\beta \mathcal{S}_n{}^\gamma\,, \\ [D, \mathcal{Q}_{n\alpha}] &= \tfrac{1}{2}\mathcal{Q}_{n\alpha}\,, \qquad & [D, \mathcal{S}_n{}^\alpha] &= -\tfrac{1}{2}\mathcal{S}_n{}^\alpha\,, \end{aligned} \tag{2.10}$$

[4] $E^{+\pm}_{mn}/E^{-\pm}_{mn}$ correspond to the positive/negative roots $\mathrm{e}_m \pm \mathrm{e}_n/{-\mathrm{e}_m} \pm \mathrm{e}_n$, $m < n$, where $\mathrm{e}_n, \mathrm{e}_m$ are usual $\mathbb{R}^N$ orthonormal vectors. A linearly independent basis for raising/lowering operators is $\{E^{+-}_{n\,n+1}, E^{++}_{N-1\,N}\}/\{E^{-+}_{n\,n+1}, E^{--}_{N-1\,N}\}$, $n = 1, \dots, N-1$, corresponding to the positive/negative simple roots.

along with identical equations for $\mathcal{Q}_{n\alpha}, \mathcal{S}_n{}^\alpha \to \bar{\mathcal{Q}}_{n\alpha}, \bar{\mathcal{S}}_n{}^\alpha$. The non-zero anti-commutators among the supercharges and their superconformal extensions are,

$$\begin{aligned}
&\{\mathcal{Q}_{m\alpha}, \bar{\mathcal{Q}}_{n\beta}\} = 2\delta_{mn} P_{\alpha\beta}\,, \qquad \{\mathcal{S}_m{}^\alpha, \bar{\mathcal{S}}_n{}^\beta\} = 2\delta_{mn} K^{\alpha\beta}\,,\\
&\{\mathcal{Q}_{m\alpha}, \bar{\mathcal{S}}_m{}^\beta\} = 2i\big(M_\alpha{}^\beta + \delta_\alpha{}^\beta D - \delta_\alpha{}^\beta H_m\big)\,,\\
&\{\bar{\mathcal{Q}}_{m\alpha}, \mathcal{S}_m{}^\beta\} = 2i\big(M_\alpha{}^\beta + \delta_\alpha{}^\beta D + \delta_\alpha{}^\beta H_m\big)\,,\\
&\{\mathcal{Q}_{m\alpha}, \mathcal{S}_n{}^\beta\} = \delta_\alpha{}^\beta E^{++}_{mn}\,, \qquad \{\bar{\mathcal{Q}}_{m\alpha}, \bar{\mathcal{S}}_n{}^\beta\} = \delta_\alpha{}^\beta E^{--}_{mn}\,,\\
&\{\mathcal{Q}_{m\alpha}, \bar{\mathcal{S}}_n{}^\beta\} = \delta_\alpha{}^\beta E^{+-}_{mn}\,, \qquad \{\bar{\mathcal{Q}}_{m\alpha}, \mathcal{S}_n{}^\beta\} = \delta_\alpha{}^\beta E^{-+}_{mn}\,,
\end{aligned} \tag{2.11}$$

where $m \neq n$ in the last two lines. Finally, under the action of the generators (2.7) the non-zero commutators are

$$\begin{aligned}
&[H_m, \mathcal{Q}_{n\alpha}] = \delta_{mn}\,\mathcal{Q}_{n\alpha}\,, && [H_m, \mathcal{S}_n{}^\alpha] = \delta_{mn}\,\mathcal{S}_n{}^\alpha\,,\\
&[H_m, \bar{\mathcal{Q}}_{n\alpha}] = -\delta_{mn}\,\bar{\mathcal{Q}}_{n\alpha}\,, && [H_m, \bar{\mathcal{S}}_n{}^\alpha] = -\delta_{mn}\,\bar{\mathcal{S}}_n{}^\alpha\,,\\
&[E^{-+}_{lm}, \mathcal{Q}_{n\alpha}] = -[E^{+-}_{ml}, \mathcal{Q}_{n\alpha}] = 2i\,\delta_{ln}\mathcal{Q}_{m\alpha}\,, && [E^{-+}_{lm}, \mathcal{S}_n{}^\alpha] = -[E^{+-}_{ml}, \mathcal{S}_n{}^\alpha] = 2i\,\delta_{ln}\mathcal{S}_m{}^\alpha\,,\\
&[E^{--}_{lm}, \mathcal{Q}_{n\alpha}] = 2i(\delta_{ln}\bar{\mathcal{Q}}_{m\alpha} - \delta_{mn}\bar{\mathcal{Q}}_{l\alpha})\,, && [E^{--}_{lm}, \mathcal{S}_n{}^\alpha] = 2i(\delta_{ln}\bar{\mathcal{S}}_m{}^\alpha - \delta_{mn}\bar{\mathcal{S}}_l{}^\alpha)\,,\\
&[E^{+-}_{lm}, \bar{\mathcal{Q}}_{n\alpha}] = -[E^{-+}_{ml}, \bar{\mathcal{Q}}_{n\alpha}] = 2i\,\delta_{ln}\bar{\mathcal{Q}}_{m\alpha}\,, && [E^{+-}_{lm}, \bar{\mathcal{S}}_n{}^\alpha] = -[E^{-+}_{ml}, \bar{\mathcal{S}}_n{}^\alpha] = 2i\,\delta_{ln}\bar{\mathcal{S}}_m{}^\alpha\,,\\
&[E^{++}_{lm}, \bar{\mathcal{Q}}_{n\alpha}] = 2i(\delta_{ln}\mathcal{Q}_{m\alpha} - \delta_{mn}\mathcal{Q}_{l\alpha})\,, && [E^{++}_{lm}, \bar{\mathcal{S}}_n{}^\alpha] = 2i(\delta_{ln}\mathcal{S}_m{}^\alpha - \delta_{mn}\mathcal{S}_l{}^\alpha)\,.
\end{aligned} \tag{2.12}$$

Suppressing spinor indices, the action of the linearly independent set of lowering operators on $\mathcal{Q}_n$ may be expressed by the following diagram.

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathcal{Q}_N & & & & & & \\
 & & & & & {\scriptstyle E^{-+}_{N-1\,N}}\nearrow & & \searrow{\scriptstyle E^{--}_{N-1\,N}} & & & & & \\
\mathcal{Q}_1 \xrightarrow[E^{-+}_{12}]{} \mathcal{Q}_2 \xrightarrow[E^{-+}_{23}]{} \mathcal{Q}_3 & \longrightarrow \cdots \longrightarrow & \mathcal{Q}_{N-1} & & & & & & \bar{\mathcal{Q}}_{N-1} & \longrightarrow \cdots \longrightarrow & \bar{\mathcal{Q}}_3 \xrightarrow[E^{-+}_{23}]{} \bar{\mathcal{Q}}_2 \xrightarrow[E^{-+}_{12}]{} \bar{\mathcal{Q}}_1 \\
 & & & & & {\scriptstyle E^{--}_{N-1\,N}}\searrow & & \nearrow{\scriptstyle E^{-+}_{N-1\,N}} & & & & & \\
 & & & & & & \bar{\mathcal{Q}}_N & & & & & &
\end{array} \tag{2.13}$$

Of course, an identical diagram applies to $\mathcal{S}_n$, $\bar{\mathcal{S}}_n$. (This diagram is helpful later for discussing the shortening conditions and the computation of superconformal characters.)

The spin generators in terms of $SU(2)$ generators may be expressed as follows,

$$[M_\alpha{}^\beta] = \begin{pmatrix} J_3 & J_+ \\ J_- & -J_3 \end{pmatrix}\,, \qquad [J_+, J_-] = 2J_3\,, \qquad [J_3, J_\pm] = \pm J_\pm\,, \tag{2.14}$$

where $(\mathcal{Q}_{n1}, \mathcal{Q}_{n2})$, $(\bar{\mathcal{Q}}_{n1}, \bar{\mathcal{Q}}_{n2})$, $(\mathcal{S}_n{}^2, -\mathcal{S}_n{}^1)$, $(\bar{\mathcal{S}}_n{}^2, -\bar{\mathcal{S}}_n{}^1)$ transform as usual spin $\frac{1}{2}$ doublets, each with J_3 eigenvalues $(\frac{1}{2}, -\frac{1}{2})$.

3. Shortening Conditions for Unitary Multiplets

For physical applications, we require states with positive real conformal dimensions, non-negative half integer spin eigenvalues and in finite dimensional irreducible representations of the R-symmetry group $SO(2N)$. Hence it is sufficient to consider superconformal representations defined by highest weight states $|\Delta, j, \mathrm{r}\rangle^{\text{h.w.}}$ with,

$$
\begin{aligned}
&(K^{\alpha\beta},\ \mathcal{S}_n{}^{\alpha},\ \bar{\mathcal{S}}_n{}^{\alpha},\ J_+,\ E^{+\pm}_{mn})|\Delta, j, \mathrm{r}\rangle^{\text{h.w.}} = 0\,, \qquad 1 \le m < n \le N\,,\\
&(D,\ J_3,\ H_m)|\Delta, j, \mathrm{r}\rangle^{\text{h.w.}} = (\Delta,\ j,\ r_m)|\Delta, j, \mathrm{r}\rangle^{\text{h.w.}}\,,\\
&\mathrm{r} = (r_1, \dots, r_N)\,,
\end{aligned}
\tag{3.1}
$$

where Δ is the conformal dimension, $j \in \frac{1}{2}\mathbb{N}$ is the spin, and with $SO(2N)$ Dynkin labels expressed in terms of $r_j \in \frac{1}{2}\mathbb{Z}$ required to satisfy,

$$
[r_1 - r_2, \dots, r_{N-2} - r_{N-1}, r_{N-1} + r_N, r_{N-1} - r_N] \in \mathbb{N}^N\,, \tag{3.2}
$$

so that, in particular, $r_1 \ge r_2 \ge \ldots \ge r_{N-1} \ge |r_N|$.

For a representation space with basis $V_{(\Delta;j;\mathrm{r})}$, the states are given by

$$
V_{(\Delta;j;\mathrm{r})} = \left\{ \prod_{\substack{\alpha,\beta,\gamma,\delta=1,2\\ \gamma\le\delta,\, 1\le n,m,r,s\le N}} (\mathcal{Q}_{n\alpha})^{\kappa_{n\alpha}} (\bar{\mathcal{Q}}_{m\beta})^{\bar{\kappa}_{m\beta}} (P_{\gamma\delta})^{k_{\gamma\delta}} (J_-)^K (E^{-\pm}_{rs})^{K_{rs}} |\Delta, j, \mathrm{r}\rangle^{\text{h.w.}} \right\}, \tag{3.3}
$$

for $\kappa_{n\alpha}, \bar{\kappa}_{m\beta} \in \{0,1\}$ and $k_{\gamma\delta}, K, K_{rs} \in \mathbb{N}$.

Unitarity requires that [14],[5]

$$
\Delta \ge \begin{cases} r_1 + j + 1\,, & \text{for } j > 0 \\ r_1\,, & \text{for } j = 0 \end{cases} \,. \tag{3.4}
$$

The superconformal multiplets may be truncated by various shortening conditions which are considered now. For the BPS shortening condition,

$$
\mathcal{Q}_{n\alpha}|\Delta, j, \mathrm{r}\rangle^{\text{h.w.}} = 0\,, \qquad \alpha = 1, 2\,, \tag{3.5}
$$

[5] In the basis (2.3), (2.4), (2.5), (2.6), all the generators of $Osp(2N|4)$ are hermitian, apart from D and $M_\alpha{}^\beta$ which are anti-hermitian. In order to impose the physically necessary unitarity conditions arising from a scalar product defined by the two point function for the conformal fields, it is sufficient to perform a similarity transformation, see [10] for example, where, in particular, D and $M_\alpha{}^\beta$ become hermitian so that $M_\alpha{}^\beta$ generates $SO(3)$, rather than $SO(2,1)$, and D has real eigenvalues which are required to be positive except for the trivial representation. Of course, such a similarity transformation does not affect the shortening conditions derived here.

then this leads to the following equations, using (2.11),

$$(E^{+\pm}_{nm},\ J_\pm,\ D \pm J_3 - H_n)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0\,, \qquad 1 \le m \le N\,, \qquad m \neq n\,, \tag{3.6}$$

so that, using (2.8), (2.14) and (3.1),

$$\Delta = r_n\,, \qquad r_1 = r_2 = \ldots = r_n\,, \qquad j = 0\,. \tag{3.7}$$

Clearly, (3.5) with (3.6) implies that $\mathcal{Q}_{m\alpha}|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0$, $m < n$.

Imposing the constraint,

$$\bar{\mathcal{Q}}_{N\alpha}|\Delta, 0, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0\,, \qquad \alpha = 1, 2\,, \tag{3.8}$$

leads to, using (2.11),

$$(E^{\pm-}_{mN},\ J_\pm,\ D \pm J_3 + H_N)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0\,, \qquad 1 \le m < N\,, \tag{3.9}$$

so that, using (2.8), (2.14) and (3.1),

$$\Delta = -r_N\,, \qquad r_1 = r_2 = \ldots = -r_N\,, \qquad j = 0\,. \tag{3.10}$$

We may also consider the semi-short multiplet condition,

$$\Big(\mathcal{Q}_{n2} - \frac{1}{2j}\mathcal{Q}_{n1}J_-\Big)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0\,, \tag{3.11}$$

whereby applying $\bar{\mathcal{S}}_l{}^2$ and using (2.11) we obtain,

$$(E^{+\pm}_{nm},\ D - J_3 - H_n - \tfrac{1}{2j}\,J_+J_-)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0\,, \qquad 1 \le m \le N\,, \qquad m \neq n\,, \tag{3.12}$$

so that, using (2.8), (2.14) and (3.1),

$$\Delta = r_n + j + 1\,, \qquad r_1 = r_2 = \ldots = r_n\,. \tag{3.13}$$

We may also consider,

$$\Big(\bar{\mathcal{Q}}_{N2} - \frac{1}{2j}\bar{\mathcal{Q}}_{N1}J_-\Big)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0\,, \tag{3.14}$$

whereby applying $\mathcal{S}_l{}^2$ and using (2.11) then

$$(E^{\pm-}_{mN},\ D - J_3 + H_N - \tfrac{1}{2j}J_+J_-)|\Delta, j, \mathrm{r}\rangle^{\mathrm{h.w.}} = 0\,, \qquad 1 \le m < N\,, \tag{3.15}$$

so that, using (2.8), (2.14) and (3.1),

$$\Delta = -r_N + j + 1\,, \qquad r_1 = r_2 = \ldots = -r_N\,. \tag{3.16}$$

Imposing both,

$$\Big(\mathcal{Q}_{N2} - \frac{1}{2j}\,\mathcal{Q}_{N1}J_-\Big)|\Delta,j,\mathrm{r}\rangle^{\mathrm{h.w.}} = 0\,, \qquad \Big(\bar{\mathcal{Q}}_{N2} - \frac{1}{2j}\,\bar{\mathcal{Q}}_{N1}J_-\Big)|\Delta,j,\mathrm{r}\rangle^{\mathrm{h.w.}} = 0\,, \tag{3.17}$$

leads to a conservation condition on the highest weight state, using (2.11),

$$\Big((2j-1)\big(2j\,P_{22} - 2\,P_{12}J_-\big) + P_{11}J_-{}^2\Big)|\Delta,j,\mathrm{r}\rangle^{\mathrm{h.w.}} = 0\,, \tag{3.18}$$

and also to, using (3.13), for $n = N$, and (3.16),

$$\Delta = j + 1\,, \qquad \mathrm{r} = 0\,. \tag{3.19}$$

It is easy to see that imposing BPS or semi-shortening conditions, other than the above, using other $\bar{\mathcal{Q}}_{n\alpha}$, $n \neq N$, leads to violations of (3.4), implying that corresponding multiplets are non-unitary. These are not considered here.

For the truncated supermultiplets $\mathcal{M}$, the Verma modules $V_{(\Delta;j;\mathrm{r})} \to \mathcal{V}^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}$ are generated by a subset of the generators in (3.3) so that it is sufficient to set some $\kappa_{n\alpha}, \bar\kappa_{m\beta}, k_{\gamma\delta}$ to zero.

Using the information encapsulated in (2.13), then the condition (3.5), for BPS multiplets, entails omitting $\mathcal{Q}_{j\alpha}$, $j = 1,\ldots,n$ from (3.3), so that $\kappa_{j\alpha} = 0$. Similarly, for (3.8), then $\kappa_{j\alpha}, \bar\kappa_{N\alpha} = 0$, $j = 1,\ldots,N{-}1$, while for the semi-shortening conditions, (3.11), $\kappa_{j2} = 0$, $j = 1,\ldots,n$, and (3.14), $\kappa_{j2}, \bar\kappa_{N2} = 0$, $j = 1,\ldots,N{-}1$. Corresponding to (3.17) with (3.18) then $\kappa_{j2}, \bar\kappa_{N2} = 0$, $j = 1,\ldots,N$, $k_{22} = 0$ for the multiplet of conserved currents. This information along with notation is summarised in the table below.

Table 1

Type	Δ	r	Omitted	Denoted
Long	$\geq r_1 + j + 1$	$r_1 \geq \ldots \geq \lvert r_N\rvert$	None	$(N, A, 0)$
Semi-Short	$r_1 + j + 1$	$r_1 = \ldots = r_n$	$\{\mathcal{Q}_{i2}\}_{i=1}^{n}$	(N, A, n) for $n < N$
Semi-Short	$r_1 + j + 1$	$r_1 = \ldots = r_N$	$\{\mathcal{Q}_{i2}\}_{i=1}^{N}$	$(N, A, +)$
Semi-Short	$r_1 + j + 1$	$r_1 = \ldots = -r_N$	$\{\mathcal{Q}_{i2}, \bar{\mathcal{Q}}_{N2}\}_{i=1}^{N-1}$	$(N, A, -)$
BPS	r_1	$r_1 = \ldots = r_n$	$\{\mathcal{Q}_{i\alpha}\}_{i=1}^{n}$	(N, B, n) for $n < N$
$\frac{1}{2}$ BPS	r_1	$r_1 = \ldots = r_N$	$\{\mathcal{Q}_{i\alpha}\}_{i=1}^{N}$	$(N, B, +)$
$\frac{1}{2}$ BPS	r_1	$r_1 = \ldots = -r_N$	$\{\mathcal{Q}_{i\alpha}, \bar{\mathcal{Q}}_{N\alpha}\}_{i=1}^{N-1}$	$(N, B, -)$
cons. current	$j + 1$	$r_i = 0$	$\{\mathcal{Q}_{i2}, \bar{\mathcal{Q}}_{N2}, P_{22}\}_{i=1}^{N}$	$(N, \mathrm{cons.})$

4. Superconformal Characters for $SO(2N)$ R-symmetry

A procedure for computing conformal characters for higher than two dimensions and $\mathcal{N}=4$ superconformal characters for four dimensions has been explained in detail elsewhere [15,16]. The procedure is also closely related to that in [10] for constructing supermultiplets by employing the Racah-Speiser algorithm. We proceed by analogy with [15,16].

Introducing variables s, x, $\mathrm{y}=(y_1,\ldots,y_N)$, we may write the character corresponding to the restricted Verma module, (3.3) for $V_{(\Delta;j;\mathrm{r})}\to\mathcal{V}^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}$, as a formal trace,

$$
\begin{aligned}
C^{\mathcal{M}}_{(\Delta;j,\mathrm{r})}(s,x,\mathrm{y}) &= \widetilde{\mathrm{Tr}}_{\mathcal{V}^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}}\big(s^{2D}x^{2J_3}{y_1}^{H_1}\cdots{y_N}^{H_N}\big)\\
&= s^{2\Delta}\,C_{2j}(x)\,C^{(N)}_{\mathrm{r}}(\mathrm{y})\\
&\quad\times\sum_{k_{\gamma\delta}\in\mathbb{N}}(s^2x^2)^{k_{11}}s^{2k_{12}}(s^2x^{-2})^{k_{22}}\\
&\quad\times\sum_{\kappa_{n\alpha},\bar{\kappa}_{m\beta}\in\{0,1\}}(s\,y_n\,x)^{\kappa_{n1}}(s\,y_n\,x^{-1})^{\kappa_{n2}}(s\,{y_m}^{-1}x)^{\bar{\kappa}_{m1}}(s\,{y_m}^{-1}x^{-1})^{\bar{\kappa}_{m2}}\,,
\end{aligned}
\tag{4.1}
$$

where the sum over $k_{\gamma\delta}$ gives the contributions of $P_{\gamma\delta}$ and that over $\kappa_{n\alpha},\bar{\kappa}_{m\beta}$ gives those of $\mathcal{Q}_{n\alpha}$, $\bar{\mathcal{Q}}^{\beta}_{m}$, and where,

$$
\begin{aligned}
C_j(x) &= \frac{x^{j+1}}{x-x^{-1}}\,,\\
C^{(N)}_{\mathrm{r}}(\mathrm{y}) &= \textstyle\prod_{j=1}^{N}{y_j}^{r_j+j-1}/\Delta(\mathrm{y}+\mathrm{y}^{-1})\,,\qquad \Delta(\mathrm{y})=\displaystyle\prod_{1\le i<j\le N}(y_i-y_j)\,,
\end{aligned}
\tag{4.2}
$$

are the Verma module characters for $SU(2)$ and $SO(2N)$, giving contributions from the J_-, $E^{-\pm}_{rs}$ generators, and the highest weight state, in (3.3).

Once the correct generators are omitted from (3.3), so that various $\kappa_{n\alpha},\bar{\kappa}_{m\beta},k_{\gamma\delta}$ are zero in (4.1), the prescription for finding the characters of corresponding unitary irreducible representations $\mathcal{R}^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}$ is simply given by,[6]

$$
\chi^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}(s,x,\mathrm{y})=\mathrm{Tr}_{\mathcal{R}^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}}\big(s^{2D}\,x^{2J_3}\,{y_1}^{H_1}\cdots{y_N}^{H_N}\big)=\mathfrak{W}^{(N)}\;C^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}(s,x,\mathrm{y})\,,\tag{4.3}
$$

where $\mathfrak{W}^{(N)}=\mathfrak{W}^{\mathcal{S}_2}\mathfrak{W}^{\mathcal{S}_N\ltimes(\mathcal{S}_2)^{N-1}}$ is the Weyl symmetriser for the maximal compact subgroup of the superconformal group, $U(1)\times SU(2)\times SO(2N)$. Here $\mathcal{S}_2$ and $\mathcal{S}_N\ltimes(\mathcal{S}_2)^{N-1}$

[6] As discussed in [15,16], the action of the Weyl symmetriser on the Verma module character corresponds to quotienting out null states in the Verma module, (3.3) for $V_{(\Delta;j;\mathrm{r})}\to\mathcal{V}^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}$, to obtain the irreducible module, (3.3) for $V_{(\Delta;j;\mathrm{r})}\to\mathcal{R}^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}$.

are the Weyl symmetry groups for $SU(2)$ and $SO(2N)$ and, for some functions $f(x)$, $f(\mathrm{y}) = f(y_1,\dots,y_N)$, the action of the relevant Weyl symmetrisers is given by,

$$
\begin{aligned}
\mathfrak{W}^{\mathcal{S}_2}\, f(x) &= f(x) + f(x^{-1})\,, \\
\mathfrak{W}^{\mathcal{S}_N \ltimes (\mathcal{S}_2)^{N-1}}\, f(\mathrm{y}) &= \textstyle\sum_{\substack{\varepsilon_1,\dots,\varepsilon_N=\pm 1 \\ \prod_i \varepsilon_i = 1}} \sum_{\sigma\in\mathcal{S}_N} f({y_{\sigma(1)}}^{\varepsilon_1},\dots,{y_{\sigma(N)}}^{\varepsilon_N})\,.
\end{aligned}
\tag{4.4}
$$

It is important to realise that the resulting characters may be expanded in terms of $SU(2)\times SO(2N)$ characters using,

$$
\begin{aligned}
\chi_j(x) &= \mathfrak{W}^{\mathcal{S}_2} C_j(x) = \frac{x^{j+1}-x^{-j-1}}{x-x^{-1}}\,, \\
\chi^{(N)}_{\mathrm{r}}(\mathrm{y}) &= \mathfrak{W}^{\mathcal{S}_N \ltimes (\mathcal{S}_2)^{N-1}}\, C^{(N)}_{\mathrm{r}}(\mathrm{y}) \\
&= \big(\det\big[{y_i}^{r_j+N-j}+{y_i}^{-r_j-N+j}\big] + \det\big[{y_i}^{r_j+N-j}-{y_i}^{-r_j-N+j}\big]\big)/2\,\Delta(\mathrm{y}+\mathrm{y}^{-1})\,,
\end{aligned}
\tag{4.5}
$$

the usual Weyl character formulae for $SU(2)$ and $SO(2N)$ finite dimensional, irreducible representations.

Defining,

$$
\begin{aligned}
P(s,x) &= \frac{1}{(1-s^2)(1-s^2x^2)(1-s^2x^{-2})}\,, \\
\mathcal{Q}_n(\mathrm{y},x) &= \prod_{j=n+1}^{N}(1+y_j x)\,, \qquad \bar{\mathcal{Q}}_n(\mathrm{y},x) = \prod_{j=1}^{n}(1+{y_j}^{-1}x)\;,
\end{aligned}
\tag{4.6}
$$

then this prescription leads to the following character formulae for the unitary irreducible representations, using (4.1) with (4.3) and with the notation of Table 1,

$$
\begin{aligned}
&\chi^{(N,i,n)}_{(\Delta;j;r_1,\dots,r_1,r_{n+1},\dots,r_N)}(s,x,\mathrm{y}) = \mathfrak{W}^{(N)}\, C^{(N,i,n)}_{(\Delta;j;r_1,\dots,r_1,r_{n+1},\dots,r_N)}(s,x,\mathrm{y}) \\
&= s^{2\Delta} P(s,x)\,\mathfrak{W}^{(N)}\Big(C_{2j}(x)C^{(N)}_{\mathrm{r}}(\mathrm{y})\,\mathcal{R}^{(N,i,n)}(s,x,\mathrm{y})\textstyle\prod_{\varepsilon=\pm1}\bar{\mathcal{Q}}_N(s^{-1}\mathrm{y},x^{\varepsilon})\Big)\,, \quad n<N\,, \\
&\chi^{(N,i,\pm)}_{(\Delta;j;r,\dots,r,\pm r)}(s,x,\mathrm{y}) = \mathfrak{W}^{(N)}\, C^{(N,i,\pm)}_{(r+j+1;j;r,\dots,r,\pm r)}(s,x,\mathrm{y}) \\
&= s^{2\Delta} P(s,x)\,\mathfrak{W}^{(N)}\Big(C_{2j}(x)C^{(N)}_{\mathrm{r}}(\mathrm{y})\,\mathcal{R}^{(N,i,\pm)}(s,x,\mathrm{y})\textstyle\prod_{\varepsilon=\pm1}\bar{\mathcal{Q}}_{N-1}(s^{-1}\mathrm{y},x^{\varepsilon})\Big)\,,
\end{aligned}
\tag{4.7}
$$

where,

$$
\begin{aligned}
\mathcal{R}^{(N,i,n)}(s,x,\mathrm{y}) &= \begin{cases} \mathcal{Q}_0(s\mathrm{y},x)\mathcal{Q}_n(s\mathrm{y},x^{-1}) & \text{for } i=A, \\ \prod_{\varepsilon=\pm1}\mathcal{Q}_n(s\mathrm{y},x^{\varepsilon}) & \text{for } i=B, \end{cases} \\
\mathcal{R}^{(N,i,\pm)}(s,x,\mathrm{y}) &= \begin{cases} \mathcal{Q}_0(s\mathrm{y},x)(1+s{y_N}^{-1}x)(1+s{y_N}^{\mp1}x^{-1}) & \text{for } i=A, \\ \prod_{\varepsilon=\pm1}(1+s{y_N}^{\mp1}x^{\varepsilon}) & \text{for } i=B, \end{cases}
\end{aligned}
\tag{4.8}
$$

and appropriate Δ, j are as given in Table 1. (The conserved current multiplet is discussed separately below.)

Using invariance of $\prod_{\varepsilon=\pm 1} \mathcal{Q}_0(s\mathrm{y}, x^\varepsilon)\bar{\mathcal{Q}}_N(s^{-1}\mathrm{y}, x^\varepsilon)$ under $\mathfrak{W}^{(N)}$ then the long multiplet character is given by,

$$\begin{aligned}\chi^{(N,\mathrm{long})}_{(\Delta;j;\mathrm{r})}(s,x,\mathrm{y}) &= \chi^{(N,A,0)}_{(\Delta;j;\mathrm{r})}(s,x,\mathrm{y}) \\ &= s^{2\Delta}P(s,x)\chi_{2j}(x)\chi^{(N)}_{\mathrm{r}}(\mathrm{y})\textstyle\prod_{\varepsilon=\pm 1}\mathcal{Q}_0(s\mathrm{y},x^\varepsilon)\bar{\mathcal{Q}}_N(s^{-1}\mathrm{y},x^\varepsilon)\,.\end{aligned} \tag{4.9}$$

This may be expanded in terms of $SU(2)\times SO(2N)$ characters using the identities,

$$P(s,x) = \frac{1}{1-s^4}\sum_{n=0}^{\infty} s^{2n}\,\chi_{2n}(x)\,, \tag{4.10}$$

along with, for later use,

$$\begin{aligned}&\textstyle\prod_{j=1}^{N}(1+t\,y_j)(1+t\,y_i{}^{-1}) \\ &= \textstyle\sum_{n=0}^{N-1}\big(t^n + t^{2N-n}\big)\chi^{(N)}_{(1^n,0^{N-n})}(\mathrm{y}) + t^N\,\chi^{(N)}_{(1^N)}(\mathrm{y}) + t^N\,\chi^{(N)}_{(1^{N-1},-1)}(\mathrm{y})\,.\end{aligned} \tag{4.11}$$

Simplification of BPS Multiplet Characters

Half BPS characters may be simplified by first writing, easily obtained from (4.7),

$$\begin{aligned}&\chi^{(N,B,\pm)}_{(r;0;r,\dots,r,\pm r)}(s,x,\mathrm{y}) \\ &= s^{2r}P(s,x)\sum_{a_1,\dots,a_N=0}^{2} s^{a_1+\dots+a_N}\chi_{j_{a_1}}(x)\cdots\chi_{j_{a_N}}(x)\chi^{(N)}_{(r-a_1,r-a_2,\dots,\pm r\mp a_N)}(\mathrm{y})\,,\end{aligned} \tag{4.12}$$

where, $j_a = \frac{1}{2}(1-(-1)^a)$ so that

$$j_0 = j_2 = 0\,, \qquad j_1 = 1\,. \tag{4.13}$$

Some further manipulation shows that (4.12) may be simlified further to,[7]

$$
\begin{aligned}
&\chi^{(N,B,\pm)}_{(r;0;r,\ldots,r,\pm r)}(s,x,\mathrm{y}) \\
&= s^{2r}P(s,x) \sum_{0\leq a_1\leq\ldots\leq a_N\leq 2} s^{a_1+\ldots+a_N}\chi_{j_{a_1,\ldots,a_N}}(x)\chi^{(N)}_{(r-a_1,r-a_2,\ldots,\pm r\mp a_N)}(\mathrm{y})\,,
\end{aligned}
\tag{4.14}
$$

where

$$
j_{a_1,\ldots,a_N} = \tfrac{1}{2}(N-(-1)^{a_1}-\ldots-(-1)^{a_N})\,. \tag{4.15}
$$

Further simplifications to the half BPS character (4.14) occur for $r<2$. For $r=1$ then,

$$
\begin{aligned}
&\chi^{(N,B,\pm)}_{(1;0;1,\ldots,1,\pm 1)}(s,x,\mathrm{y}) \\
&= \mathcal{A}_{(1,0)}(s,x)\ \chi^{(N)}_{(1^{N-1},\pm 1)}(\mathrm{y}) + \mathcal{A}_{(\frac{3}{2},\frac{1}{2})}(s,x)\ \chi^{(N)}_{(1^{N-1},0)}(\mathrm{y}) + \mathcal{A}_{(2,0)}(s,x)\ \chi^{(N)}_{(1^{N-1},\mp 1)}(\mathrm{y}) \\
&\quad + \textstyle\sum_{n=0}^{N-2}\mathcal{D}_{\frac{1}{2}(N-n)}(s,x)\ \chi^{(N)}_{(1^n,0^{N-n})}(\mathrm{y})\,,
\end{aligned}
\tag{4.16}
$$

where,

$$
\mathcal{A}_{(\Delta,j)} = s^{2\Delta}\chi_{2j}(x)P(s,x)\,, \qquad \mathcal{D}_j = s^{2j+2}\Big(\chi_{2j}(x)-s^2\chi_{2j-2}(x)\Big)P(s,x)\,, \tag{4.17}
$$

are the characters for unitary irreducible representations of the conformal group in three dimensions, $SO(3,2)$, [18] - see also [15] - whereby $\mathcal{A}_{(\Delta,j)}$ corresponds to an unconstrained spin j field, conformal dimension $\Delta\geq j=1$, while $\mathcal{D}_j$ corresponds to a conserved current with spin j, conformal dimension $j+1$, including all their conformal descendants (or derivatives acting on fields).

Similarly, for $r=\frac{1}{2}$ then,

$$
\chi^{(N,B,\pm)}_{(\frac{1}{2};0;\frac{1}{2},\ldots,\frac{1}{2},\pm\frac{1}{2})}(s,x,\mathrm{y}) = \mathcal{D}_{\mathrm{Rac}}(s,x)\chi^{(N)}_{(\frac{1}{2},\ldots,\frac{1}{2},\pm\frac{1}{2})}(\mathrm{y}) + \mathcal{D}_{\mathrm{Di}}(s,x)\chi^{(N)}_{(\frac{1}{2},\ldots,\frac{1}{2},\mp\frac{1}{2})}(\mathrm{y}) \tag{4.18}
$$

---

[7] Another consistency check is for $N=4$, or $SO(8)$ R-symmetry, whereby this formula leads directly to,

$$
\begin{aligned}
\chi^{(4,B,-)}_{(r;0;r,r,r,-r)}(s,x,\mathrm{y}) &= s^{2r}P(s,x)\Big(\chi^{(4)}_{(r,r,r,-r)}(\mathrm{y}) + s\chi_1(x)\chi^{(4)}_{(r,r,r,1-r)}(\mathrm{y}) + s^2\chi^{(4)}_{(r,r,r,2-r)}(\mathrm{y}) \\
&+ s^2\chi_2(x)\chi^{(4)}_{(r,r,r-1,1-r)}(\mathrm{y}) + s^3\chi_1(x)\chi^{(4)}_{(r,r,r-1,2-r)}(\mathrm{y}) + s^3\chi_3(x)\chi^{(4)}_{(r,r-1,r-1,1-r)}(\mathrm{y}) \\
&+ s^4\chi^{(4)}_{(r,r,r-2,2-r)}(\mathrm{y}) + s^4\chi_2(x)\chi^{(4)}_{(r,r-1,r-1,2-r)}(\mathrm{y}) + s^4\chi_4(x)\chi^{(4)}_{(r-1,r-1,r-1,1-r)}(\mathrm{y}) \\
&+ s^5\chi_1(x)\chi^{(4)}_{(r,r-1,r-2,2-r)}(\mathrm{y}) + s^5\chi_3(x)\chi^{(4)}_{(r-1,r-1,r-1,2-r)}(\mathrm{y}) + s^6\chi^{(4)}_{(r,r-2,r-2,2-r)}(\mathrm{y}) \\
&+ s^6\chi_2(x)\chi^{(4)}_{(r-1,r-1,r-2,2-r)}(\mathrm{y}) + s^7\chi_1(x)\chi^{(4)}_{(r-1,r-2,r-2,2-r)}(\mathrm{y}) + s^8\chi^{(4)}_{(r-2,r-2,r-2,2-r)}(\mathrm{y})\Big)
\end{aligned}
$$

which corresponds exactly to the graviton spectrum derived in [17].

where,

$$\mathcal{D}_{\mathrm{Rac}}(s,x) = \frac{s+s^3}{(1-s^2x^2)(1-s^2x^{-2})}\,, \qquad \mathcal{D}_{\mathrm{Di}}(s,x) = \frac{s^2(x+x^{-1})}{(1-s^2x^2)(1-s^2x^{-2})}\,, \tag{4.19}$$

are characters for the free field representations of $SO(3,2)$, the so called 'Di', respectively, 'Rac', singleton representations [19], corresponding to a free spin $\frac{1}{2}$, respectively, scalar, field with conformal dimension 1, respectively, $\frac{1}{2}$, and all its descendants.

Finally, from (4.14) it may be shown that,

$$\chi^{(N,B,\pm)}_{(0;0;0,\ldots,0)}(s,x,\mathrm{y}) = 1\,, \tag{4.20}$$

the character for the identity representation.

For other BPS characters, from (4.7), we may write, similarly to (4.12) with (4.13),

$$\begin{aligned}
&\chi^{(N,B,n)}_{(r_1;0;r_1,r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}(s,x,\mathrm{y}) \\
&= s^{2r_1}P(s,x)\sum_{\substack{a_1,\ldots,a_N\\ \bar{a}_{n+1},\ldots,\bar{a}_N}=0}^{2} s^{a_1+\ldots+a_N+\bar{a}_{n+1}+\ldots+\bar{a}_N}\textstyle\prod_{i=1}^{N}\chi_{j_{a_i}}(x)\prod_{i=n+1}^{N}\chi_{j_{\bar{a}_i}}(x) \\
&\qquad\qquad \times \chi^{(N)}_{(r_1-a_1,\ldots,r_1-a_n,r_{n+1}+\bar{a}_{n+1}-a_{n+1},\ldots,r_N+\bar{a}_N-a_N)}(\mathrm{y})\,,
\end{aligned} \tag{4.21}$$

and, similarly to (4.14) with (4.15), using that $r_1 = r_2 = \ldots = r_n$,

$$\begin{aligned}
&\chi^{(N,B,n)}_{(r_1;0;r_1,r_1,\ldots,r_1,r_{n+1},\ldots,r_N)}(s,x,\mathrm{y}) \\
&= s^{2r_1}P(s,x) \\
&\times \sum_{0\le a_1\le\ldots\le a_n\le 2}\ \sum_{\substack{a_{n+1},\ldots,a_N\\ \bar{a}_{n+1},\ldots,\bar{a}_N}=0}^{2} s^{a_1+\ldots+a_N+\bar{a}_{n+1}+\ldots+\bar{a}_N}\chi_{j_{a_1,a_2,\ldots,a_n}}(x)\textstyle\prod_{i=n+1}^{N}\chi_{j_{a_i}}(x)\chi_{j_{\bar{a}_i}}(x) \\
&\qquad\qquad \times \chi^{(N)}_{(r_1-a_1,\ldots,r_1-a_n,r_{n+1}+\bar{a}_{n+1}-a_{n+1},\ldots,r_N+\bar{a}_N-a_N)}(\mathrm{y})\,,
\end{aligned} \tag{4.22}$$

where,

$$j_{a_1,\ldots,a_n} = \tfrac{1}{2}(n-(-1)^{a_1}-\ldots-(-1)^{a_n})\,. \tag{4.23}$$

Without further restrictions on r_i, or s, x, y, there appear to be no further simplifications to (4.22).

The Conserved Current Multiplet Character

The conserved current multiplet (which has been excluded so far from the above discussion) corresponding to the semi-shortening condition (3.17), has character,

$$\begin{aligned}
&\chi^{(N,\mathrm{cons.})}_{(j+1;j;0,\ldots,0)}(s,x,\mathrm{y}) \\
&= s^{2j+2}P(s,x)\mathfrak{W}^{(N)}\Big(C_{2j}(x)C^{(N)}_0(\mathrm{y})(1-s^2x^{-2})\mathcal{Q}_0(s\mathrm{y},x)\bar{\mathcal{Q}}_N(s^{-1}\mathrm{y},x)\bar{\mathcal{Q}}_{N-1}(s^{-1}\mathrm{y},x^{-1})\Big)\,,
\end{aligned} \tag{4.24}$$

which ensures that the contributions from the supercharges $\mathcal{Q}_{n2}$, $n=1,\dots,N$, $\bar{\mathcal{Q}}_{N2}$ along with P_{22} are omitted from the Verma module character (4.1).

To simplify, it is useful to observe that $\mathcal{Q}_0(s\mathrm{y},x)\bar{\mathcal{Q}}_N(s^{-1}\mathrm{y},x)$ is invariant under action by $\mathfrak{W}^{\mathcal{S}_N\ltimes(\mathcal{S}_2)^{N-1}}$ and, further, that $\mathfrak{W}^{\mathcal{S}_N\ltimes(\mathcal{S}_2)^{N-1}}C_0(\mathrm{y})\bar{\mathcal{Q}}_{N-1}(\mathrm{y},t)=1$ so that

$$\begin{aligned}\chi^{(N,\text{cons.})}_{(j+1;j;0,\dots,0)}(s,x,\mathrm{y}) &= s^{2j+2}P(s,x)\mathfrak{W}^{\mathcal{S}_2}\Big(C_{2j}(x)(1-s^2x^{-2})\mathcal{Q}_0(s\mathrm{y},x)\bar{\mathcal{Q}}_N(s^{-1}\mathrm{y},x)\Big)\\ &= \sum_{n=0}^{N-1}\Big(\mathcal{D}_{j+\frac12 n}(s,x)+\mathcal{D}_{j+N-\frac12 n}(s,x)\Big)\chi^{(N)}_{(1^n,0^{N-n})}(\mathrm{y})\\ &\qquad + \mathcal{D}_{j+\frac12 N}(s,x)\Big(\chi^{(N)}_{(1^N)}(\mathrm{y})+\chi^{(N)}_{(1^{N-1},-1)}(\mathrm{y})\Big)\,,\end{aligned} \tag{4.25}$$

using (4.11), to expand in x, and subsequently the expression for the conserved current character in (4.17).

Long Multiplet Decompositions

Using linearity of $\mathfrak{W}^{(N)}$ we may easily obtain, for $j\geq\frac12$, $r_1>r_2$,

$$\begin{aligned}&\chi^{(N,\text{long})}_{(r_1+j+1;j;r_1,r_2,\dots,r_N)}(s,x,\mathrm{y})\\ &\qquad = \mathfrak{W}^{(N)}(1+s\,y_1\,x^{-1})C^{(N,A,1)}_{(r_1+j+1;j;r_1,r_2,\dots,r_N)}(s,x,\mathrm{y})\\ &\qquad = \chi^{(N,A,1)}_{(r_1+j+1;j;r_1,r_2,\dots,r_N)}(s,x,\mathrm{y})+\chi^{(N,A,1)}_{(r_1+j+\frac32;j-\frac12;r_1+1,r_2,\dots,r_N)}(s,x,\mathrm{y})\,,\end{aligned} \tag{4.26}$$

which expresses the reducibility of a long multiplet with $\Delta=r_1+j+1$ into a sum of semi-short multiplets.

For semi-short multiplet characters it may be shown that, for $r_1=r_2=\ldots=r_n>r_{n+1}$, $r>0$,[8]

$$\begin{aligned}\chi^{(N,A,1)}_{(r_1+j+1;j;r_1,\dots,r_1,r_{n+1},\dots,r_N)}(s,x,\mathrm{y}) &= \chi^{(N,A,n)}_{(r_1+j+1;j;r_1,\dots,r_1,r_{n+1},\dots,r_N)}(s,x,\mathrm{y})\,,\\ \chi^{(N,A,1)}_{(r+j+1,j,r,\dots,r,\pm r)}(s,x,\mathrm{y}) &= \chi^{(N,A,\pm)}_{(r+j+1,j,r,\dots,r,\pm r)}(s,x,\mathrm{y})\,.\end{aligned} \tag{4.27}$$

---

[8] This employs,

$$\begin{aligned}&\chi^{(N,A,1)}_{(r_1+j+1;j;r_1,\dots,r_1,r_{n+1},\dots,r_N)}(s,x,\mathrm{y})\\ &\qquad = \mathfrak{W}^{(N)}\textstyle\prod_{i=2}^n(1+s\,y_i\,x^{-1})C^{(N,A,n)}_{(r_1+j+1;j;r_1,\dots,r_1,r_{n+1},\dots,r_N)}(s,x,\mathrm{y})\,,\end{aligned}$$

along with the identity $\chi^{(N)}_{\mathrm{r}^w}(\mathrm{y})=(-1)^{|w|}\chi^{(N)}_{\mathrm{r}}(\mathrm{y})$, $w\in\mathcal{S}_N\ltimes(\mathcal{S}_2)^{N-1}$, where $\mathrm{r}^w=w(\mathrm{r}+\rho)-\rho$, with $w\in\mathcal{S}_N\ltimes(\mathcal{S}_2)^{N-1}$, the Weyl group, and $\rho=(N{-}1,N{-}2,\dots,0)$, being the Weyl vector. In particular, $\chi^{(N)}_{(r_1,\dots,r_1,r_1+1,\dots,r_N)}(\mathrm{y})=0$ which implies $\chi^{(N,A,n)}_{(\Delta';j';r_1,\dots,r_1,r_1+1,\dots,r_N)}(s,x,\mathrm{y})=0$. The $(N,A,\pm)$ cases may be found similarly.

Similarly, for the conserved current multiplet character,

$$\chi^{(N,A,1)}_{(j+1;j;0,\dots,0)}(s,x,\mathrm{y}) = \chi^{(N,\mathrm{cons.})}_{(j+1;j;0,\dots,0)}(s,x,\mathrm{y})\,. \tag{4.28}$$

In order to avoid ambiguity, it is assumed in what follows that $r_n > r_{n+1}$, for (N,A,n) semishort multiplet characters, unless otherwise stated.

From (4.26) with (4.27) and (4.28), then we have, for $j \geq \frac{1}{2}$, $r_1 > r_{n+1}$, $n < N$,

$$\begin{aligned}&\chi^{(N,\mathrm{long})}_{(r_1+j+1;j;r_1,r_1,\dots,r_1,r_{n+1},\dots,r_N)}(s,x,\mathrm{y})\\ &= \chi^{(N,A,n)}_{(r_1+j+1;j;r_1,\dots,r_1,r_{n+1},\dots,r_N)}(s,x,\mathrm{y}) + \chi^{(N,A,1)}_{(r_1+j+\frac{3}{2};j-\frac{1}{2};r_1+1,r_1,\dots,r_1,r_{n+1},\dots,r_N)}(s,x,\mathrm{y})\,,\end{aligned} \tag{4.29}$$

along with, for $r > 0$,

$$\begin{aligned}&\chi^{(N,\mathrm{long})}_{(r+j+1;j;r,\dots,r,\pm r)}(s,x,\mathrm{y})\\ &= \chi^{(N,A,\pm)}_{(r+j+1;j;r,\dots,r,\pm r)}(s,x,\mathrm{y}) + \chi^{(N,A,1)}_{(r_1+j+\frac{3}{2};j-\frac{1}{2};r+1,r,\dots,r,\pm r)}(s,x,\mathrm{y})\,,\end{aligned} \tag{4.30}$$

and,

$$\chi^{(N,\mathrm{long})}_{(j+1;j;0,\dots,0)}(s,x,\mathrm{y}) = \chi^{(N,\mathrm{cons.})}_{(j+1;j;0,\dots,0)}(s,x,\mathrm{y}) + \chi^{(N,A,1)}_{(j+\frac{3}{2};j-\frac{1}{2};1,0,\dots,0)}(s,x,\mathrm{y})\,. \tag{4.31}$$

It is also easily seen that,

$$\begin{aligned}\chi^{(N,A,1)}_{(r_1+\frac{1}{2};-\frac{1}{2};r_1,r_2,\dots,r_N)}(s,x,\mathrm{y}) &= \mathfrak{W}^{(N)}(1+s\,y_1\,x)C^{(N,B,1)}_{(r_1+\frac{1}{2};-\frac{1}{2};r_1,r_2,\dots,r_N)}(s,x,\mathrm{y})\\ &= \chi^{(N,B,1)}_{(r_1+1;0;r_1+1,r_2,\dots,r_N)}(s,x,\mathrm{y})\,,\end{aligned} \tag{4.32}$$

using $\mathfrak{W}^{\mathcal{S}_2}C_{-1}(x) = 0$, $\mathfrak{W}^{\mathcal{S}_2}xC_{-1}(x) = \mathfrak{W}^{\mathcal{S}_2}C_0(x) = 1$, so that, for $r_1 > r_2$,

$$\begin{aligned}\chi^{(N,\mathrm{long})}_{(r_1+1;0;r_1,r_2,\dots,r_N)}(s,x,\mathrm{y}) &= \mathfrak{W}^{(N)}(1+s\,y_1\,x^{-1})C^{(N,A,1)}_{(r_1+1;0;r_1,r_2,\dots,r_N)}(s,x,\mathrm{y})\\ &= \chi^{(N,A,1)}_{(r_1+1;0;r_1,r_2,\dots,r_N)}(s,x,\mathrm{y}) + \chi^{(N,B,1)}_{(r_1+2;0;r_1+2,r_2,\dots,r_N)}(s,x,\mathrm{y})\,,\end{aligned} \tag{4.33}$$

which expresses the reducibility of a long multiplet with $\Delta = r_1+1$ into a sum of a $(N,B,1)$ BPS and a semi-short multiplet.

Thus, from (4.33) with (4.27) and (4.28), we have, for $r_1 = r_n > r_{n+1}$,

$$\begin{aligned}&\chi^{(N,\mathrm{long})}_{(r_1+1;0;r_1,\dots,r_1,r_{n+1},\dots,r_N)}(s,x,\mathrm{y})\\ &= \chi^{(N,A,n)}_{(r_1+1;0;r_1,\dots,r_1,r_{n+1},\dots,r_N)}(s,x,\mathrm{y}) + \chi^{(N,B,1)}_{(r_1+2;0;r_1+2,r_1,\dots,r_1,r_{n+1},\dots,r_N)}(s,x,\mathrm{y})\,,\end{aligned} \tag{4.34}$$

along with, for $r > 0$,

$$
\begin{aligned}
&\chi^{(N,\mathrm{long})}_{(r+1;0;r,\dots,r,\pm r)}(s,x,\mathrm{y}) \\
&= \chi^{(N,A,\pm)}_{(r+1;0;r,\dots,r,\pm r)}(s,x,\mathrm{y}) + \chi^{(N,B,1)}_{(r+2;0;r+2,r,\dots,r,\pm r)}(s,x,\mathrm{y})\,,
\end{aligned}
\tag{4.35}
$$

and,

$$
\chi^{(N,\mathrm{long})}_{(1;0;0,\dots,0)}(s,x,\mathrm{y}) = \chi^{(N,\mathrm{cons.})}_{(1;0;0,\dots,0)}(s,x,\mathrm{y}) + \chi^{(N,B,1)}_{(2;0;2,0,\dots,0)}(s,x,\mathrm{y})\,. \tag{4.36}
$$

(4.26), (4.29), (4.30), (4.31), (4.33), (4.34), (4.35), (4.36) appear to exhaust all possibilities for long multiplet decompositions (and are also consistent with limits, as discussed in section 5) and have important consequences for any superconformal field theory in three dimensions. In particular, they imply that all $(N,B,\pm)$ and (N,B,n), $N>n>1$, short multiplet operators as well as certain $(N,B,1)$ short multiplet operators in $\mathcal{R}_{(r_1+1,r_2,\dots,r_N)}$, $r_1 \geq r_2$, $SO(2N)$ R-symmetry representations must remain protected against gaining anomalous dimensions. The decomposition formulae (4.26), (4.33) have essentially appeared in [9] where also comments about the protectedness of certain operators in three dimensional superconformal field theories were made. (Note that all the decomposition formulae here are consequences of the basic formula (4.26). This is not surprising as a similar thing happens for decomposition formulae of long multiplets for $\mathcal{N}=4$ superconformal symmetry in four dimensions [15].)

5. Reduction to Subalgebra Characters

Here are described certain limits that can be taken in the previous BPS and semi-short multiplet characters that isolate contributions from fewer states in each multiplet and hence lead to significant simplifications. These limits are equivalent to reductions of the characters to those for various subgroups of the superconformal group, as explained in more detail in appendix A.

BPS Limits

The $U(1)\otimes SO(2N-2m)$ Sector

By considering,

$$
\begin{aligned}
&\chi^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}(\delta u^{\frac12}, x, (\delta^{-2}u)^{\frac1m}\hat{\mathrm{y}}, \bar{\mathrm{y}}) \\
&\quad = \mathrm{Tr}_{\mathcal{R}^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}}(\delta^{2\mathcal{H}_m}\, u^{\mathcal{I}_m}\, x^{2J_3}\, \hat{y}_1{}^{H_1}\cdots \hat{y}_m{}^{H_m}\, \bar{y}_1{}^{H_{m+1}}\cdots \bar{y}_{N-m}{}^{H_N})\,, \\
&\hat{\mathrm{y}} = (\delta^{-2}u)^{-\frac1m}(y_1,\dots,y_m)\,, \qquad \textstyle\prod_{\hat{m}=1}^m \hat{y}_{\hat{m}} = 1\,, \qquad \bar{\mathrm{y}} = (y_{m+1},\dots,y_N)\,, \\
&\mathcal{H}_m = D - \tfrac1m \textstyle\sum_{\hat{m}=1}^m H_m\,, \qquad \mathcal{I}_m = D + \tfrac1m \textstyle\sum_{\hat{m}=1}^m H_m\,,
\end{aligned}
\tag{5.1}
$$

in the limit $\delta \to 0$, it is clear that only those states in $\mathcal{R}^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}$ for which $\mathcal{H}_m$ has zero eigenvalue contribute. In particular, this applies to the highest weight state in the (N,B,m) BPS multiplet.

It can be shown that,

$$\lim_{\delta\to 0} \chi^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}(\delta u^{\frac{1}{2}}, x, (\delta^{-2}u)^{\frac{1}{m}}\hat{\mathrm{y}}, \bar{\mathrm{y}}) = \chi^{U(1)\otimes SO(2N-2m)}_{(R;\bar{\mathrm{r}})}(u,\bar{\mathrm{y}}) = u^R \chi^{(N-m)}_{\bar{\mathrm{r}}}(\bar{\mathrm{y}})\,, \tag{5.2}$$

in terms of $U(1)_{\mathcal{I}_m}\otimes SO(2N-2m)$ characters, for appropriate R, $\bar{\mathrm{r}}$, see appendix A.

An identity which is useful for simplifying the limits considered here is the following, for $\mathrm{r} = (r_1, r_2, \ldots, r_N)$,

$$\begin{aligned} &\chi^{(N)}_{\mathrm{r}}(\delta^{-\frac{2}{m}}\hat{\mathrm{u}}, \bar{\mathrm{y}}) \sim \delta^{-\frac{2}{m}(r_1+\ldots+r_m)} s_{\hat{\mathrm{r}}}(\hat{\mathrm{u}})\, \chi^{(N-m)}_{\bar{\mathrm{r}}}(\bar{\mathrm{y}})\,, \\ &\hat{\mathrm{r}} = (r_1,\ldots,r_m)\,, \qquad \bar{\mathrm{r}} = (r_{m+1},\ldots,r_N)\,, \end{aligned} \tag{5.3}$$

giving the leading behaviour as $\delta \to 0$, where $s_{\hat{\mathrm{r}}}(\hat{\mathrm{u}})$ is a Schur polynomial,[9]

$$s_{\hat{\mathrm{r}}}(\hat{\mathrm{u}}) = \det\big[\hat{u}_i{}^{r_j+n-j}\big]/\Delta(\hat{\mathrm{u}})\,, \qquad s_{(r,\ldots,r)}(\hat{\mathrm{u}}) = \textstyle\prod_{i=1}^m \hat{u}_i{}^r\,. \tag{5.4}$$

Using (5.3), we may obtain from (4.7) that the limit taken in (5.1) gives the following $U(1)\otimes SO(2N-2m)$ characters, for $n \geq m$, see appendix A,

$$\begin{aligned} \chi^{U(1)\otimes SO(2N-2m)}_{(r_1;\bar{\mathrm{r}})}(u,\bar{\mathrm{y}}) &= \lim_{\delta\to 0}\chi^{(N,B,n)}_{(r_1;0;\mathrm{r})}(\delta u^{\frac{1}{2}}, x, (\delta^{-2}u)^{\frac{1}{m}}\hat{\mathrm{y}}, \bar{\mathrm{y}}) = u^{2r_1}\chi^{(N-m)}_{\bar{\mathrm{r}}}(\bar{\mathrm{y}})\,, \\ \chi^{U(1)\otimes SO(2N-2m)}_{(r;r,\ldots,r,\pm r)}(u,\bar{\mathrm{y}}) &= \lim_{\delta\to 0}\chi^{(N,B,\pm)}_{(r;0;r,\ldots,r,\pm r)}(\delta u^{\frac{1}{2}}, x, (\delta^{-2}u)^{\frac{1}{m}}\hat{\mathrm{y}}, \bar{\mathrm{y}}) = u^{2r}\chi^{(N-m)}_{(r,\ldots,r,\pm r)}(\bar{\mathrm{y}})\,. \end{aligned} \tag{5.5}$$

For $m=1$, we have in addition to (5.5) that,

$$\chi^{U(1)\otimes SO(2N-2)}_{(r_1+2;\bar{\mathrm{r}})}(u,\bar{\mathrm{y}}) = \lim_{\delta\to 0}\chi^{(N,\mathrm{long})}_{(r_1+1;0;\mathrm{r})}(\delta u^{\frac{1}{2}}, x, \delta^{-2}u, \bar{\mathrm{y}})\,, \tag{5.6}$$

---

[9] The identity (5.3) may be obtained by noting that, for small δ,

$$\begin{aligned} C^{(N)}_{\mathrm{r}}(\delta^{-\frac{2}{m}}\hat{\mathrm{u}}, \bar{\mathrm{y}}) &\sim \widetilde{C}^{(m)}_{\hat{\mathrm{r}}}(\delta^{-\frac{2}{m}}\hat{\mathrm{u}})\, C^{(N-m)}_{\bar{\mathrm{r}}}(\bar{\mathrm{y}})\,, \qquad \widetilde{C}^{(m)}_{\hat{\mathrm{r}}}(\hat{\mathrm{u}}) = \textstyle\prod_{i=1}^m \hat{u}_i{}^{\hat{r}_i+m-i}/\Delta(\hat{\mathrm{u}})\,, \\ \mathfrak{W}^{\mathcal{S}_N\ltimes(\mathcal{S}_2)^{N-1}} C^{(N)}_{\mathrm{r}}(\delta^{-\frac{2}{m}}\hat{\mathrm{u}}, \bar{\mathrm{y}}) &\sim \mathfrak{W}^{\mathcal{S}_m}\,\mathfrak{W}^{\mathcal{S}_{(N-m)}\ltimes(\mathcal{S}_2)^{N-m-1}} \widetilde{C}^{(m)}_{\hat{\mathrm{r}}}(\delta^{-\frac{2}{m}}\hat{\mathrm{u}})\, C^{(N-m)}_{\bar{\mathrm{r}}}(\bar{\mathrm{y}})\,. \end{aligned}$$

Here, $\widetilde{C}^{(m)}_{\hat{\mathrm{r}}}(\hat{\mathrm{u}})$ is equivalent to the $U(m)$ Verma module character while $\mathfrak{W}^{\mathcal{S}_m}$ is the Weyl symmetriser for $U(m)$, acting on $\hat{\mathrm{u}}$, so that for any $f(\hat{\mathrm{u}}) = f(\hat{u}_1,\ldots,\hat{u}_m)$, $\mathfrak{W}^{\mathcal{S}_m}(\hat{\mathrm{u}}) = \sum_{\sigma\in\mathcal{S}_m} f(\hat{u}_{\sigma(1)},\ldots,\hat{u}_{\sigma(m)})$. Hence, $\mathfrak{W}^{\mathcal{S}_m}\widetilde{C}^{(m)}_{\hat{\mathrm{r}}}(\hat{\mathrm{u}}) = s_{\hat{\mathrm{r}}}(\hat{\mathrm{u}})$, the $U(m)$ Weyl character. Here also, $\mathfrak{W}^{\mathcal{S}_{N-m}\ltimes(\mathcal{S}_2)^{N-m-1}}$ acts on $\bar{\mathrm{y}}$, so that $\mathfrak{W}^{\mathcal{S}_{N-m}\ltimes(\mathcal{S}_2)^{N-m-1}} C^{(N-m)}_{\bar{\mathrm{r}}}(\bar{\mathrm{y}}) = \chi^{(N-m)}_{\bar{\mathrm{r}}}(\bar{\mathrm{y}})$.

while other long multiplet characters, for $\Delta \geq r_1+j+1$, $j \neq 0$, along with semi-short multiplet characters vanish, consistent with (4.33). Since, in the limit taken in (5.2), long multiplet characters vanish for $m > 1$, this provides further evidence, apart from long multiplet decomposition formulae listed in the previous section, that (N, B, m), $m > 1$ BPS operators must remain protected in any three dimensional superconformal field theory.

The $U(1)$ Sectors

Similarly, by considering,

$$\begin{aligned}
&\chi^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}(\delta u^{\frac{1}{2}}, x, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_1 \dots, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm\frac{1}{N}}\hat{y}_N{}^{\pm 1}) \\
&= \mathrm{Tr}_{\mathcal{R}^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}}(\delta^{2\mathcal{H}_\pm}\, u^{\mathcal{I}_\pm}\, x^{2J_3}\, \hat{y}_1{}^{H_1} \cdots \hat{y}_N{}^{\pm H_N}) , \\
&\mathcal{H}_\pm = D - \tfrac{1}{N}(H_1 + \ldots + H_{N-1} \pm H_N) , \quad \mathcal{I}_\pm = D + \tfrac{1}{N}(H_1 + \ldots + H_{N-1} \pm H_N) ,
\end{aligned} \tag{5.7}$$

separately in the limit $\delta \to 0$, isolating states for which $\mathcal{H}_\pm$ has zero eigenvalues, only the corresponding $\frac{1}{2}$ BPS character is non-zero giving,

$$\begin{aligned}
\chi^{U(1)}_{(r)}(u) &= \lim_{\delta\to 0} \chi^{(N,B,\pm)}_{(r;0;r,\dots,r,\pm r)}(\delta u^{\frac{1}{2}}, x, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_1 \dots, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm\frac{1}{N}}\hat{y}_N{}^{\pm 1}) \\
&= u^{2r} .
\end{aligned} \tag{5.8}$$

Semi-short Limits

The $U(1) \otimes Osp(2N{-}2m|2)$ Sector

Considering, for $\hat{\mathrm{y}}$, $\bar{\mathrm{y}}$ as in (5.1),

$$\begin{aligned}
&\chi^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}(\delta\bar{s}^{\frac{1}{2}}, \delta^{-1}\bar{s}^{\frac{1}{2}}, (\delta^{-2}u)^{\frac{1}{m}}\hat{\mathrm{y}}, \bar{\mathrm{y}}) \\
&= \mathrm{Tr}_{\mathcal{R}^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}}(\delta^{2\mathcal{J}_m}\, u^{\mathcal{K}_m}\, \bar{s}^{\bar{D}}\, \hat{y}_1{}^{H_1} \cdots \hat{y}_m{}^{H_m}\, \bar{y}_1{}^{H_{m+1}} \cdots \bar{y}_{N-m}{}^{H_N}) , \\
&\mathcal{J}_m = D - J_3 - \tfrac{1}{m}\textstyle\sum_{\hat{m}=1}^{m} H_m , \quad \mathcal{K}_m = \tfrac{1}{m}\textstyle\sum_{\hat{m}=1}^{m} H_m , \qquad \bar{D} = D + J_3 ,
\end{aligned} \tag{5.9}$$

in the limit $\delta \to 0$, again, it is clear that only those states in $\mathcal{R}^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}$ for which the eigenvalue of $\mathcal{J}_m$ is zero contribute.

It can be shown that,

$$\begin{aligned}
\lim_{\delta\to 0} \chi^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}(\delta\bar{s}^{\frac{1}{2}}, \delta^{-1}\bar{s}^{\frac{1}{2}}, (\delta^{-2}u)^{\frac{1}{m}}\hat{\mathrm{y}}, \bar{\mathrm{y}}) &= \chi^{(U(1)\otimes Osp(2N-2m|2),i)}_{(R;\bar{\Delta};\bar{\mathrm{r}})}(u, \bar{s}, \bar{\mathrm{y}}) \\
&= u^R \chi^{(Osp(2N-2m|2),i)}_{(\bar{\Delta},\bar{\mathrm{r}})}(\bar{s}, \bar{\mathrm{y}}) ,
\end{aligned} \tag{5.10}$$

in terms of $U(1)_{\mathcal{K}_m} \otimes Osp(2N{-}2m|2)$ characters, for appropriate i, R, $\bar{\Delta}$, see appendix A for notation.

Similar to above, using (4.7) with (5.3), and also,

$$\chi_{2j}(\delta^{-1}\bar{s}^{\frac{1}{2}}) \sim \delta^{-2j}\,\bar{s}^{j}\,, \tag{5.11}$$

giving the leading behaviour as $\delta \to 0$, then we have, for semi-short and conserved multiplet cases, for $n \geq m$ and with $\bar{\mathrm{r}}$ as in (5.3),

$$\begin{aligned}
\chi^{(U(1)\otimes Osp(2N-2m|2),\mathrm{long})}_{(r_1+1;r_1+2j+m+1;\bar{\mathrm{r}})}(u,\bar{s},\bar{\mathrm{y}}) &= \lim_{\delta\to 0}\chi^{(N,A,n)}_{(r_1+j+1;j;\mathrm{r})}(\delta\bar{s}^{\frac{1}{2}},\delta^{-1}\bar{s}^{\frac{1}{2}},(\delta^{-2}u)^{\frac{1}{m}}\hat{\mathrm{y}},\bar{\mathrm{y}}) \\
&= u^{r_1+1}\,\frac{\bar{s}^{r_1+2j+m+1}}{1-\bar{s}^2}\;\chi^{(N-m)}_{\bar{\mathrm{r}}}(\bar{\mathrm{y}})\prod\nolimits_{\varepsilon=\pm 1}\prod\nolimits_{i=1}^{N-m}(1+\bar{y}_i{}^{\varepsilon}\bar{s})\,, \\
\chi^{(U(1)\otimes Osp(2N-2m|2),\mathrm{long})}_{(r+1;r+2j+m+1;r,\dots,r,\pm r)}(u,\bar{s},\bar{\mathrm{y}}) &= \lim_{\delta\to 0}\chi^{(N,A,\pm)}_{(r+j+1;j;r,\dots,r,\pm r)}(\delta\bar{s}^{\frac{1}{2}},\delta^{-1}\bar{s}^{\frac{1}{2}},(\delta^{-2}u)^{\frac{1}{m}}\hat{\mathrm{y}},\bar{\mathrm{y}}) \\
&= u^{r+1}\,\frac{\bar{s}^{r+2j+m+1}}{1-\bar{s}^2}\;\chi^{(N-m)}_{(r,\dots,r,\pm r)}(\bar{\mathrm{y}})\prod\nolimits_{\varepsilon=\pm 1}\prod\nolimits_{i=1}^{N-m}(1+\bar{y}_i{}^{\varepsilon}\bar{s})\,, \\
\chi^{(U(1)\otimes Osp(2N-2m|2),\mathrm{long})}_{(1;2j+m+1;0,\dots,0)}(u,\bar{s},\bar{\mathrm{y}}) &= \lim_{\delta\to 0}\chi^{(N,\mathrm{cons.})}_{(j+1;j;0,\dots,0)}(\delta\bar{s}^{\frac{1}{2}},\delta^{-1}\bar{s}^{\frac{1}{2}},(\delta^{-2}u)^{\frac{1}{m}}\hat{\mathrm{y}},\bar{\mathrm{y}}) \\
&= u\,\frac{\bar{s}^{2j+m+1}}{1-\bar{s}^2}\prod\nolimits_{\varepsilon=\pm 1}\prod\nolimits_{i=1}^{N-m}(1+\bar{y}_i{}^{\varepsilon}\bar{s})\,.
\end{aligned} \tag{5.12}$$

For (N,B,n) BPS multiplet characters there are a number of cases to consider. For $n<m$ we have, for $r_n=\ldots=r_1$ and $r_{n+1}=\ldots=r_m=r_1-1\geq r_{m+1}\geq\ldots\geq|r_N|$,

$$\begin{aligned}
&\chi^{(U(1)\otimes Osp(2N-2m|2),\mathrm{long})}_{(r_1;r_1+m-n;\bar{\mathrm{r}})}(u,\bar{s},\bar{\mathrm{y}}) = \lim_{\delta\to 0}\chi^{(N,B,n)}_{(r_1;0;\mathrm{r})}(\delta\bar{s}^{\frac{1}{2}},\delta^{-1}\bar{s}^{\frac{1}{2}},(\delta^{-2}u)^{\frac{1}{m}}\hat{\mathrm{y}},\bar{\mathrm{y}}) \\
&= u^{r_1}\frac{\bar{s}^{r_1+m-n}}{1-\bar{s}^2}\;\chi^{(N-m)}_{\bar{\mathrm{r}}}(\bar{\mathrm{y}})\prod\nolimits_{\varepsilon=\pm 1}\prod\nolimits_{i=1}^{N-m}(1+\bar{y}_i{}^{\varepsilon}\,\bar{s})\,,
\end{aligned} \tag{5.13}$$

while for $n=m$, so that $r_1=\ldots=r_n>r_{n+1}\geq\ldots\geq|r_N|$,

$$\begin{aligned}
\chi^{(U(1)\otimes Osp(2N-2m|2),\mathrm{long})}_{(r_1;r_1;\bar{\mathrm{r}})}(u,\bar{s},\bar{\mathrm{y}}) &= \lim_{\delta\to 0}\chi^{(N,B,m)}_{(r_1;0;\mathrm{r})}(\delta\bar{s}^{\frac{1}{2}},\delta^{-1}\bar{s}^{\frac{1}{2}},(\delta^{-2}u)^{\frac{1}{m}}\hat{\mathrm{y}},\bar{\mathrm{y}}) \\
&= u^{r_1}\frac{\bar{s}^{r_1}}{1-\bar{s}^2}\,\chi^{(N-m)}_{\bar{\mathrm{r}}}(\bar{\mathrm{y}})\prod\nolimits_{\varepsilon=\pm 1}\prod\nolimits_{i=1}^{N-m}(1+\bar{y}_i{}^{\varepsilon}\,\bar{s})\,.
\end{aligned} \tag{5.14}$$

For $n>m$ we have, using results from appendix A,

$$\begin{aligned}
&\chi^{(U(1)\otimes Osp(2N-2m|2),n)}_{(r_1;r_1;\bar{\mathrm{r}})}(u,\bar{s},\bar{\mathrm{y}}) = \lim_{\delta\to 0}\chi^{(N,B,n)}_{(r_1;0;\mathrm{r})}(\delta\bar{s}^{\frac{1}{2}},\delta^{-1}\bar{s}^{\frac{1}{2}},(\delta^{-2}u)^{\frac{1}{m}}\hat{\mathrm{y}},\bar{\mathrm{y}}) \\
&= u^{r_1}\frac{\bar{s}^{r_1}}{1-\bar{s}^2}\;\mathfrak{W}^{\mathcal{S}_{N-m}\ltimes(\mathcal{S}_2)^{N-m-1}}\Big(C^{(N-m)}_{\bar{\mathrm{r}}}(\bar{\mathrm{y}})\prod\nolimits_{i=n-m+1}^{N-m}(1+\bar{y}_i\,\bar{s})\prod\nolimits_{i=1}^{N-m}(1+\bar{y}_i{}^{-1}\bar{s})\Big)\,,
\end{aligned} \tag{5.15}$$

so that

$$
\begin{aligned}
&\chi^{(U(1)\otimes Osp(2N-2m|2),n)}_{(r_1;r_1;\bar{\mathrm{r}})}(u,\bar s,\bar{\mathrm{y}})\\
&= u^{r_1}\,\frac{\bar s^{r_1}}{1-\bar s^2}\sum_{0\le a_1\le\ldots\le a_{n-m}\le 1}\ \sum^{1}_{\substack{a_{n-m+1},\ldots,a_{N-m}\\ \bar a_{n-m+1},\ldots,\bar a_{N-m}}=0} s^{a_1+\ldots+a_{N-m}+\bar a_{n-m+1}+\ldots+\bar a_{N-m}}\\
&\qquad\qquad\qquad \times\chi^{(N-m)}_{(r_1-a_1,\ldots,r_1-a_{n-m},r_{n+1}+\bar a_{n-m+1}-a_{n-m+1},\ldots,r_N+\bar a_{N-m}-a_{N-m})}(\bar{\mathrm{y}})\,.
\end{aligned}\tag{5.16}
$$

Finally, for the $\frac{1}{2}$ BPS cases,

$$
\begin{aligned}
\chi^{(U(1)\otimes Osp(2N-2m|2),\pm)}_{(r;r;r,\ldots,r,\pm r)}(u,\bar s,\bar{\mathrm{y}}) &= \lim_{\delta\to 0}\chi^{(N,B,\pm)}_{(r;0;r,\ldots,r,\pm r)}(\delta\bar s^{\frac12},\delta^{-1}\bar s^{\frac12},(\delta^{-2}u)^{\frac1m}\hat{\mathrm{y}},\bar{\mathrm{y}})\\
= u^r\,\frac{\bar s^r}{1-\bar s^2}\sum_{0\le a_1\le\ldots\le a_{N-m}\le 1} & \bar s^{a_1+\ldots+a_{N-m}}\chi^{(N-m)}_{(r-a_1,\ldots,r-a_{N-m-1},\pm r\mp a_{N-m})}(\bar{\mathrm{y}})\,.
\end{aligned}\tag{5.17}
$$

In particular, this implies,

$$
\begin{aligned}
\chi^{(U(1)\otimes Osp(2N-2m|2),\pm)}_{(\frac12;\frac12;\frac12,\ldots,\frac12,\pm\frac12)}(u,\bar s,\bar{\mathrm{y}}) &= \lim_{\delta\to 0}\chi^{(N,B,\pm)}_{(\frac12;0;\frac12,\ldots,\frac12,\pm\frac12)}(\delta\bar s^{\frac12},\delta^{-1}\bar s^{\frac12},(\delta^{-2}u)^{\frac1m}\hat{\mathrm{y}},\bar{\mathrm{y}})\\
&= u^{\frac12}\,\frac{\bar s^{\frac12}}{1-\bar s^2}\Big(\chi^{(N-m)}_{(\frac12,\ldots,\frac12,\pm\frac12)}(\bar{\mathrm{y}})+\bar s\,\chi^{(N-m)}_{(\frac12,\ldots,\frac12,\mp\frac12)}(\bar{\mathrm{y}})\Big)\,,
\end{aligned}\tag{5.18}
$$

agreeing with (4.18), in this limit, using (5.3) with (5.4) for $r=\frac{1}{2}$.

For $m>1$, we have for long multiplet characters,

$$
\begin{aligned}
&\chi^{(U(1)\otimes Osp(2N-2m|2),\mathrm{long})}_{(r_1+1;r_1+2j+m+1;r_{m+1},\ldots,r_N)}(u,\bar s,\bar{\mathrm{y}})\\
&= \lim_{\delta\to 0}\chi^{(N,\mathrm{long})}_{(r_1+j+1;j;r_1,\ldots,r_1,r_{m+1},\ldots,r_N)}(\delta\bar s^{\frac12},\delta^{-1}\bar s^{\frac12},(\delta^{-2}u)^{\frac1m}\hat{\mathrm{y}},\bar{\mathrm{y}})\,,
\end{aligned}\tag{5.19}
$$

which is in accord with (4.26) with (4.27). For $m=1$, we have that

$$
(1+u)\chi^{(U(1)\otimes Osp(2N-2|2),\mathrm{long})}_{(r_1+1;r_1+2j+2;r_2,\ldots,r_N)}(u,\bar s,\bar{\mathrm{y}}) = \lim_{\delta\to 0}\chi^{(N,\mathrm{long})}_{(r_1+j+1;j;r_1,\ldots,r_N)}(\delta\bar s^{\frac12},\delta^{-1}\bar s^{\frac12},\delta^{-2}u,\bar{\mathrm{y}})\,,\tag{5.20}
$$

compatible with (4.26).

The $U(1)\otimes SU(1,1)$ Sectors

Similarly, by considering,

$$
\begin{aligned}
&\chi^{\mathcal M}_{(\Delta;j;\mathrm{r})}(\delta\bar s^{\frac12},\delta^{-1}\bar s^{\frac12},(\delta^{-2}u)^{\frac1N}\hat y_1\ldots,(\delta^{-2}u)^{\frac1N}\hat y_{N-1},(\delta^{-2}u)^{\pm\frac1N}\hat y_N{}^{\pm1})\\
&= \mathrm{Tr}_{\mathcal{R}^{\mathcal M}_{(\Delta;j;\mathrm{r})}}(\delta^{2\mathcal{J}_\pm}\,u^{\mathcal{K}_\pm}\,\bar s^{\bar D}\,\hat y_1{}^{H_1}\cdots\hat y_N{}^{\pm H_N})\,,\\
&\mathcal{J}_\pm = D-J_3-\tfrac{1}{N}(H_1+\ldots+H_{N-1}\pm H_N)\,,\quad \mathcal{K}_\pm=\tfrac{1}{N}(H_1+\ldots+H_{N-1}\pm H_N)\,,
\end{aligned}\tag{5.21}
$$

separately in the limit $\delta \to 0$, we have for the corresponding $(N, A, \pm)$ multiplet character,

$$
\begin{aligned}
&\chi^{U(1)\otimes SU(1,1)}_{(r+1;r+2j+1+N)}(u,\bar{s}) \\
&= \lim_{\delta\to 0} \chi^{(N,A,\pm)}_{(r+j+1;j;r,\dots,r,\pm r)}(\delta \bar{s}^{\frac{1}{2}}, \delta^{-1}\bar{s}^{\frac{1}{2}}, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_1 \dots, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm\frac{1}{N}}\hat{y}_N{}^{\pm 1}) \\
&= u^{r+1}\, \frac{\bar{s}^{r+2j+N+1}}{1-\bar{s}^2}\,,
\end{aligned}
\tag{5.22}
$$

and, for the conserved current character, in either limit in (5.21),

$$
\begin{aligned}
&\chi^{U(1)\otimes SU(1,1)}_{(1;2j+1+N)}(u,\bar{s}) \\
&= \lim_{\delta\to 0} \chi^{(N,\text{cons.})}_{(j+1;j;0,\dots,0)}(\delta \bar{s}^{\frac{1}{2}}, \delta^{-1}\bar{s}^{\frac{1}{2}}, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_1 \dots, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm\frac{1}{N}}\hat{y}_N{}^{\pm 1}) \\
&= u\, \frac{\bar{s}^{2j+N+1}}{1-\bar{s}^2}\,.
\end{aligned}
\tag{5.23}
$$

For (N, B, n), $n \le N-1$, BPS characters with $r_n = \ldots = r_1$ and $r_{n+1} = \ldots = r_N = \pm r_1 \mp 1$ and taking corresponding limit in (5.21) then,

$$
\begin{aligned}
&\chi^{U(1)\otimes SU(1,1)}_{(r_1;r_1+N-n)}(u,\bar{s}) \\
&= \lim_{\delta\to 0} \chi^{(N,B,n)}_{(r_1;0;\mathrm{r})}(\delta \bar{s}^{\frac{1}{2}}, \delta^{-1}\bar{s}^{\frac{1}{2}}, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_1 \dots, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm\frac{1}{N}}\hat{y}_N{}^{\pm 1}) \\
&= u^{r_1}\, \frac{\bar{s}^{r_1+N-n}}{1-\bar{s}^2}\,.
\end{aligned}
\tag{5.24}
$$

For the $(N, B, \pm)$ half BPS cases and in the corresponding limit in (5.21) we have,

$$
\begin{aligned}
&\chi^{U(1)\otimes SU(1,1)}_{(r;r)}(u,\bar{s}) \\
&= \lim_{\delta\to 0} \chi^{(N,B,n)}_{(r;0;r,\dots,r,\pm r)}(\delta \bar{s}^{\frac{1}{2}}, \delta^{-1}\bar{s}^{\frac{1}{2}}, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_1 \dots, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm\frac{1}{N}}\hat{y}_N{}^{\pm 1}) \\
&= u^{r}\, \frac{\bar{s}^{r}}{1-\bar{s}^2}\,,
\end{aligned}
\tag{5.25}
$$

along with,

$$
\begin{aligned}
&\chi^{U(1)\otimes SU(1,1)}_{(\frac{1}{2};\frac{3}{2})}(u,\bar{s}) \\
&= \lim_{\delta\to 0} \chi^{(N,B,n)}_{(\frac{1}{2};0;\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2})}(\delta \bar{s}^{\frac{1}{2}}, \delta^{-1}\bar{s}^{\frac{1}{2}}, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_1 \dots, (\delta^{-2}u)^{\frac{1}{N}}\hat{y}_{N-1}, (\delta^{-2}u)^{\pm\frac{1}{N}}\hat{y}_N{}^{\pm 1}) \\
&= u^{\frac{1}{2}}\, \frac{\bar{s}^{\frac{3}{2}}}{1-\bar{s}^2}\,.
\end{aligned}
\tag{5.26}
$$

All other characters, apart from the long multiplet one corresponding to (4.35), in the limit (5.21) vanish.

The Superconformal Index

The (single particle) superconformal indices [9,20] may be computed by taking the limit $u \to -1$ in the $U(1) \otimes Osp(2N{-}2|2)$ characters above, *i.e.* for $m = 1$ in the semi-short limits above.[10]

In particular, from (5.20), this limit ensures that $Osp(2N|4)$ long multiplet characters vanish, and hence do not contribute in the decomposition of partition functions, in the same limit, in terms of $Osp(2N|4)$ characters. Thus, partition functions evaluated in this limit receive contributions from protected operators only. It should be noted however that the magnitude of the numbers obtained for counting operators by expansion in terms of characters, in this limit, provide only a lower bound on the numbers of actual protected operators due to the $(-1)^F$ sign in the index, for fermion number F.

6. Partition Functions for $\mathcal{N} = 6$ Superconformal Chern-Simons Theory

In [2], a class of $\mathcal{N} = 6$ superconformal Chern Simons theories, with gauge group $U(n) \times U(n)$, was proposed. For levels $\pm k$ in the Chern Simons terms, these theories admit a dual description in terms of M theory compactified on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$.

Here, the free field partition function of the theory, where the effective 't Hooft coupling $n/k \to 0$, so that $k \to \infty$, in the large $n < k$ limit, is computed using appropriate superconformal characters.

The supergravity partition function, obtained using the duality proposed in [2] in the $n/k \to \infty$ limit, for $n > k \to \infty$, is also computed using appropriate superconformal characters.

Free Field Theory

For $\mathcal{N} = 6$ $U(n) \times U(n)$ superconformal Chern-Simons theory the dynamical field content consists of scalars ϕ_1, ϕ_2 and spin half fermions ψ_1, ψ_2 belonging to the 'Rac', respectively 'Di', representations of $SO(3,2)$, mentioned after (4.19). The fields are listed in Table 2 showing also their $SO(6)$ R-symmetry eigenvalues and gauge group representations.

[10] It is easily checked that for $N = 3$, $m = 1$, then (5.18) for $(u, \bar{s}, y_2, y_3) \to (-1, -x, y_1, y_2)$ matches with the index for fundamental fields computed in section 2 of [6].

Table 2

field	Δ	$SO(3,2)$ rep.	$SO(6)$ rep.	$U(n)\times U(n)$ rep.
ϕ_1	$\frac{1}{2}$	'Rac'	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(n,\overline{n})$
ψ_1	1	'Di'	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(n,\overline{n})$
ϕ_2	$\frac{1}{2}$	'Rac'	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(\overline{n},n)$
ψ_2	1	'Di'	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(\overline{n},n)$

Here the $SO(6)$ orthonormal basis labels (r,q,p) are related to $SU(4)$ Dynkin labels $[a,b,c]$ by,

$$(r,q,p)\rightarrow[q{+}p,r{-}q,q{-}p]\,, \tag{6.1}$$

so that $(\frac{1}{2},\frac{1}{2},\frac{1}{2})\rightarrow[1,0,0]$, $(\frac{1}{2},\frac{1}{2},-\frac{1}{2})\rightarrow[0,0,1]$.

As may be evident from (4.18), ϕ_1 and ψ_1, respectively ϕ_2 and ψ_2, belong to the half BPS multiplet $(3,B,+)$, respectively $(3,B,-)$. (In both cases, the scalar is the primary field of the multiplet.)

For free field theory, the single particle partition function is given by,

$$Y_{\rm free}(s,x,{\rm y},{\rm u},{\rm v})={\rm Tr}\Big(s^{2D}x^{2J_3}{y_1}^{H_1}{y_2}^{H_2}{y_3}^{H_3}{u_1}^{L_1}\cdots{u_n}^{L_n}{v_1}^{M_1}\cdots{v_n}^{M_n}\Big)\,, \tag{6.2}$$

where the trace is over states corresponding to ϕ_1,ϕ_2, ψ_1,ψ_2 and all their superconformal descendants, of the form (3.3), and $s,x,{\rm y},{\rm u},{\rm v}$ are 'fugacities' with $L_1,\ldots,L_n$, $M_1,\ldots,M_n$ being usual $U(n)\times U(n)$ Cartan subalgebra generators.

Thus we may write, in terms of characters,

$$Y_{\rm free}(s,x,{\rm y},{\rm u},{\rm v})=f_+(s,x,{\rm y})p_1({\rm u})p_1({\rm v}^{-1})+f_-(s,x,{\rm y})p_1({\rm u}^{-1})p_1({\rm v})\,, \tag{6.3}$$

defining, for subsequent use,

$$p_j({\rm u})=\sum_{i=1}^{n}{u_i}^j\,,\qquad p_j({\rm u}^{-1})=p_{-j}({\rm u})\,, \tag{6.4}$$

so that $p_1({\rm u})p_1({\rm v}^{-1})$, $p_1({\rm u}^{-1})p_1({\rm v})$ correspond to the characters for the $(n,\overline{n})$, respectively, $(\overline{n},n)$ representations of $U(n)\times U(n)$, and where

$$\begin{aligned}f_\pm(s,x,{\rm y})&=\chi^{(3,B,\pm)}_{(\frac{1}{2};0;\frac{1}{2},\frac{1}{2},\pm\frac{1}{2})}(s,x,{\rm y})\\&=\big(y_1\,y_2\,y_3\big)^{\mp\frac{1}{2}}\Big(\textstyle\sum_{j=1}^{3}{y_j}^{\pm1}+(y_1\,y_2\,y_3)^{\pm1}\Big)\mathcal{D}_{\rm Rac}(s,x)\\&\quad+\big(y_1\,y_2\,y_3\big)^{\pm\frac{1}{2}}\Big(\textstyle\sum_{j=1}^{3}{y_j}^{\mp1}+(y_1\,y_2\,y_3)^{\mp1}\Big)\mathcal{D}_{\rm Di}(s,x)\,,\end{aligned} \tag{6.5}$$

being half BPS characters (4.18) for $SO(6)$ R-symmetry.

The multi-particle partition function, receiving contributions from only gauge invariant operators, is given by the usual integral over the gauge group, namely,

$$Z^{(n)}_{\text{free}}(s,x,\mathrm{y}) = \int_{U(n)} \mathrm{d}\mu(\mathrm{u}) \int_{U(n)} \mathrm{d}\mu(\mathrm{v}) \exp\left(\sum_{j=1}^{\infty} \frac{1}{j} Y_{\text{free}}(s^j, (-1)^{j+1}x^j, \mathrm{y}^j, \mathrm{u}^j, \mathrm{v}^j) \right), \qquad (6.6)$$

where the signs on x take account of particle statistics.

The integral may be evaluated in the large n limit by using a method described in [21] (see also [22] for other applications). We may first expand,

$$\begin{aligned} &Z^{(n)}_{\text{free}}(s,x,\mathrm{y}) \\ &= \sum_{\underline{\lambda},\underline{\rho}} \frac{1}{z_{\underline{\lambda}} z_{\underline{\rho}}} f_{+\,\underline{\lambda}}(s,x,\mathrm{y}) f_{-\,\underline{\rho}}(s,x,\mathrm{y}) \int_{U(n)} \mathrm{d}\mu(\mathrm{u}) \int_{U(n)} \mathrm{d}\mu(\mathrm{v}) p_{\underline{\lambda}}(\mathrm{u}) p_{\underline{\rho}}(\mathrm{u}^{-1}) p_{\underline{\rho}}(\mathrm{v}) p_{\underline{\lambda}}(\mathrm{v}^{-1}) \,, \end{aligned} \qquad (6.7)$$

in terms of partitions,

$$\begin{aligned} \underline{\lambda} &= (\lambda_1,\ldots,\lambda_j,\ldots)\,, \qquad \textstyle\sum_{j\geq 1} j\lambda_j = |\underline{\lambda}| \in \mathbb{N}\,, \\ \underline{\rho} &= (\rho_1,\ldots,\rho_j,\ldots)\,, \qquad \textstyle\sum_{j\geq 1} j\rho_j = |\underline{\rho}| \in \mathbb{N}\,, \end{aligned} \qquad (6.8)$$

where, for $\underline{\sigma} = (\sigma_1,\ldots,\sigma_j,\ldots)$,

$$z_{\underline{\sigma}} = \prod_{j\geq 1} \sigma_j! j^{\sigma_j}\,, \quad f_{\pm\,\underline{\sigma}}(s,x,\mathrm{y}) = \prod_{j\geq 1} f_{\pm}(s^j, (-1)^{j+1}x^j, \mathrm{y}^j)^{\sigma_j}\,, \quad p_{\underline{\sigma}}(\mathrm{x}) = \prod_{j\geq 1} p_j(\mathrm{x})^{\sigma_j}\,. \qquad (6.9)$$

Using the orthogonality relation for power symmetric polynomials,

$$\int_{U(n)} \mathrm{d}\mu(\mathrm{u}) p_{\underline{\lambda}}(\mathrm{u}) p_{\underline{\rho}}(\mathrm{u}^{-1}) = z_{\underline{\lambda}} \delta_{\underline{\lambda}\,\underline{\rho}}\,, \qquad |\underline{\lambda}|, |\underline{\rho}| \leq n\,, \qquad (6.10)$$

then we trivially obtain in the large n limit,

$$\begin{aligned} Z^{(n)}_{\text{free}}(s,x,\mathrm{y}) \sim Z_{\text{free}}(s,x,y) &= \sum_{\underline{\lambda}} f_{+\,\underline{\lambda}}(s,x,\mathrm{y}) f_{-\,\underline{\lambda}}(s,x,\mathrm{y}) \\ &= \prod_{j\geq 1} \sum_{\lambda_j \geq 0} \big(f_+(s^j,(-1)^{j+1}x^j,\mathrm{y}^j) f_-(s^j,(-1)^{j+1}x^j,\mathrm{y}^j) \big)^{\lambda_j} \\ &= \prod_{j\geq 1} \frac{1}{1 - f_+(s^j,(-1)^{j+1}x^j,\mathrm{y}^j) f_-(s^j,(-1)^{j+1}x^j,\mathrm{y}^j)}\,. \end{aligned} \qquad (6.11)$$

This result was also derived in [6] by using saddle point methods, in the context of showing superconformal index matching.

By decomposing the product of half BPS characters (6.5),

$$f_+(s,x,\mathrm{y})f_-(s,x,\mathrm{y}) = \chi^{(3,B,2)}_{(1;0;1,1,0)}(s,x,\mathrm{y}) + \sum_{j=0}^{\infty} \chi^{(3,\mathrm{cons.})}_{(j+1;j;0,0,0)}(s,x,\mathrm{y})\,, \tag{6.12}$$

it is interesting to note that (6.11) may be obtained by considering a field theory with fields in the $(3,B,2)$ representation and conserved currents transforming in the adjoint representation of $U(m)$ for large m. For this theory, using (6.12),

$$Y(s,x,\mathrm{y},\mathrm{z}) = f_+(s,x,\mathrm{y})f_-(s,x,\mathrm{y})p_1(\mathrm{z})p_1(\mathrm{z}^{-1})\,, \tag{6.13}$$

so that the corresponding multi-particle partition function is given by,

$$Z(s,x,\mathrm{y}) = \int_{U(m)} \mathrm{d}\mu(\mathrm{z}) \exp\left(\sum_{j=1}^{\infty} \frac{1}{j} Y(s^j, (-1)^{j+1}x^j, \mathrm{y}^j, \mathrm{u}^j, \mathrm{z}^j)\right). \tag{6.14}$$

In the large m limit, it is easily seen that $Z(s,x,\mathrm{y})$ matches with (6.11). It may be interesting to understand this equivalence from a gauge theory or string theory perspective.

Supergravity Limit

In the strong coupling limit, large n/k, $n > k \to \infty$, as explained in [2], using results of [23], the single particle states (gravitons) effectively belong to scalar superconformal representations with conformal dimensions r and in the $\mathcal{R}_{(r,r,0)}$ $SO(6)$ representations, for $r \in \mathbb{N}$, $r > 0$, so that, in terms of notation here, they belong to $(3,B,2)$ BPS multiplets.

Taking account of superconformal descendants, we may thus write for the single particle partition function,

$$Y_{\text{sugra}}(s,x,\mathrm{y}) = \sum_{r=1}^{\infty} \chi^{(3,B,2)}_{(r;0;r,r,0)}(s,x,\mathrm{y})\,, \tag{6.15}$$

where using (4.7), for $\mathfrak{W}^{\mathcal{S}_3 \ltimes (\mathcal{S}_2)^2}$ acting on $\mathrm{y} = (y_1,y_2,y_3)$,

$$\chi^{(3,B,2)}_{(r;0;r,r,0)}(s,x,\mathrm{y}) = s^{2r} P(s,x) \mathfrak{W}^{\mathcal{S}_3 \ltimes (\mathcal{S}_2)^2} \left(C^{(3)}_{(r,r,0)}(\mathrm{y}) \prod_{\varepsilon=\pm 1} (1 + s y_3 x^{\varepsilon}) \prod_{j=1}^{3} (1 + s {y_i}^{-1} x^{\varepsilon}) \right). \tag{6.16}$$

Thus, using the definition of the $SO(6)$ Verma module character (4.2), for $N = 3$, and summing over r in (6.15) with (6.16), we may write,

$$\begin{aligned} &Y_{\text{sugra}}(s,x,\mathrm{y}) \\ &= s^2 P(s,x) \mathfrak{W}^{\mathcal{S}_3 \ltimes (\mathcal{S}_2)^2} \left(\frac{{y_1}^3 {y_2}^2 \prod_{\varepsilon=\pm 1}(1 + s y_3 x^{\varepsilon}) \prod_{i=1}^{3} (1 + s {y_i}^{-1} x^{\varepsilon})}{(1 - s^2 y_1 y_2) \prod_{1 \le j < k \le 3} ({y_j}^{-1} - {y_k}^{-1})(1 - y_j y_k)} \right). \end{aligned} \tag{6.17}$$

Simplifying the $\mathcal{S}_3$ part of the action of $\mathfrak{W}^{\mathcal{S}_3\ltimes(\mathcal{S}_2)^2}$ in the latter we end up with the more succinct formula,

$$\begin{aligned} &Y_{\rm sugra}(s,x,\mathrm{y}) \\ &= f(s,x,y_1,y_2,y_3) + f(s,x,\tfrac{1}{y_1},\tfrac{1}{y_2},y_3) + f(s,x,\tfrac{1}{y_1},y_2,\tfrac{1}{y_3}) + f(s,x,y_1,\tfrac{1}{y_2},\tfrac{1}{y_3}) - 1\,, \end{aligned} \tag{6.18}$$

where

$$f(s,x,y_1,y_2,y_3) = P(s,x)\frac{\prod_{\varepsilon=\pm1}(1+s^3\,y_1\,y_2\,y_3\,x^\varepsilon)\prod_{i=1}^3(1+s\,y_i x^\varepsilon)}{\prod_{1\le j<k\le 3}^3(1-s^2 y_j y_k)(1-{y_j}^{-1}{y_k}^{-1})}\,. \tag{6.19}$$

We then have the multi-particle (free graviton gas) partition function given by the usual plethystic exponential, also taking into account particle statistics,

$$Z_{\rm sugra}(s,x,\mathrm{y}) = \exp\left(\sum_{j=1}^\infty \frac{1}{j} Y_{\rm sugra}(s^j,(-1)^{j+1}x^j,{y_1}^j,{y_2}^j,{y_3}^j)\right). \tag{6.20}$$

7. Counting Multi-Trace Gauge Invariant Operators

The limits in $Osp(2N|4)$ characters discussed in section 5, giving reductions to subalgebra characters, are equivalent to decoupling limits, that isolate sectors of operators in partition functions. By taking such limits, for $N=3$, in (6.11), (6.20), and decomposing in terms of characters in the same limits we are able to count corresponding multi-trace gauge invariant operators in the free field, supergravity limits. For free field theory, the counting numbers obtained provide an upper bound on the numbers of protected operators in the interacting theory, while for the supergravity limit they are expected to count actually protected ones.

For $f_\pm(s,x,\mathrm{y}) \to f_\pm^G(u,\bar{\mathrm{y}})$, $Y_{\rm sugra}(s,x,\mathrm{y}) \to Y^G_{\rm sugra}(u,\bar{\mathrm{y}})$, in the BPS limits, where G is the corresponding subgroup, and $f_\pm(s,x,\mathrm{y}) \to f_\pm^H(u,\bar{s},\bar{\mathrm{y}})$, $Y_{\rm sugra}(s,x,\mathrm{y}) \to Y^H_{\rm sugra}(u,\bar{s},\bar{\mathrm{y}})$, in the semi-short limits, where H is the corresponding subgroup, the limits generically give,

$$\begin{aligned} f^G_\pm(u,\bar{\mathrm{y}}) &= \chi^{(3,B,\pm)}_{(\frac12;0;\frac12,\frac12,\pm\frac12)}(u,\bar{\mathrm{y}})\,, \qquad & Y^G_{\rm sugra}(u,\bar{\mathrm{y}}) &= \textstyle\sum_{r=1}^\infty \chi^{(3,B,2)}_{(r;0;r,r,0)}(u,\bar{\mathrm{y}})\,, \\ f^H_\pm(u,\bar{s},\bar{\mathrm{y}}) &= \chi^{(3,B,\pm)}_{(\frac12;0;\frac12,\frac12,\pm\frac12)}(u,\bar{s},\bar{\mathrm{y}})\,, \qquad & Y^H_{\rm sugra}(u,\bar{s},\bar{\mathrm{y}}) &= \textstyle\sum_{r=1}^\infty \chi^{(3,B,2)}_{(r;0;r,r,0)}(u,\bar{s},\bar{\mathrm{y}})\,, \end{aligned} \tag{7.1}$$

where, for the superconformal characters,

$$\chi^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}(s,x,\mathrm{y}) \to \chi^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}(u,\bar{\mathrm{y}})\,, \qquad \chi^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}(s,x,\mathrm{y}) \to \chi^{\mathcal{M}}_{(\Delta;j;\mathrm{r})}(u,\bar{s},\bar{\mathrm{y}})\,, \tag{7.2}$$

in the same limits. An important point for the corresponding multi-particle partition functions is properly taking into account particle statistics which differs in both cases. For $Z_{\rm free}(s,x,{\rm y}) \to Z^G_{\rm free}(u,\bar{\rm y})$, $Z_{\rm sugra}(s,x,{\rm y}) \to Z^G_{\rm sugra}(u,\bar{\rm y})$, in the BPS limits and $Z_{\rm free}(s,x,{\rm y}) \to Z^H_{\rm free}(u,\bar s,\bar{\rm y})$, $Z_{\rm sugra}(s,x,{\rm y}) \to Z^H_{\rm sugra}(u,\bar s,\bar{\rm y})$, in the semishort limits, consistency requires,[11]

$$
\begin{aligned}
Z^G_{\rm free}(u,\bar{\rm y}) = {}& \prod_{j=1}^{\infty} \frac{1}{1-f^G_+(u^j,\bar{\rm y}^j) f^G_-(u^j,\bar{\rm y}^j)}\,, \\
Z^G_{\rm sugra}(u,\bar{\rm y}) = {}& \exp\left(\sum_{j=1}^{\infty} \frac{1}{j} Y^G_{\rm sugra}(u^j,\bar{\rm y}^j)\right), \\
Z^H_{\rm free}(u,\bar s,\bar{\rm y}) = {}& \prod_{j=1}^{\infty} \frac{1}{1-f^H_+(u^j\alpha_j,\bar s^j/\alpha_j,\bar{\rm y}^j) f^H_-(u^j\alpha_j,\bar s^j/\alpha_j,\bar{\rm y}^j)}\bigg|_{\alpha_j=(-1)^{j+1}}\,, \\
Z^H_{\rm sugra}(u,\bar s,\bar{\rm y}) = {}& \exp\left(\sum_{j=1}^{\infty} \frac{1}{j} Y^H_{\rm sugra}(u^j\alpha_j,\bar s^j/\alpha_j,\bar{\rm y}^j)\Big|_{\alpha_j=(-1)^{j+1}}\right).
\end{aligned}
\tag{7.3}
$$

BPS Cases

The $U(1)$ Sectors

Corresponding to (5.8) we consider the $(3,B,+)$ limit, whereby

$$
\chi^{(3,B,+)}_{(r;0;r,r,\pm r)}(u) = \chi^{U(1)}_{(r)}(u) = u^{2r}, \tag{7.4}
$$

so that $f^{U(1)}_+(u) = u$, $f^{U(1)}_-(u) = 0$. It is also easily seen that $Y^{U(1)}_{\rm sugra}(u) = 0$ in the limit (5.8). Thus, from (7.3),

$$
Z^{U(1)}_{\rm free}(u) = Z^{U(1)}_{\rm sugra}(u) = 1\,, \tag{7.5}
$$

so that, clearly, there are no $(3,B,+)$ BPS gauge invariant operators in the free field or supergravity spectrum, apart from the identity operator. The same result applies to $(3,B,-)$ BPS operators.

The $U(1)\otimes U(1)$ Sector

Corresponding to (5.5) for $N=3$, $m=2$, we have that,

$$
\begin{aligned}
\chi^{(3,B,2)}_{(r;0;r,r,q)}(u,y) = \chi^{U(1)\otimes U(1)}_{(r;q)}(u,y) = u^{2r} y^q\,, \\
\chi^{(3,B,\pm)}_{(r;0;r,r,\pm r)}(u,y) = \chi^{U(1)\otimes U(1)}_{(r;\pm r)}(u,y) = u^{2r} y^{\pm r}\,,
\end{aligned}
\tag{7.6}
$$

[11] No signs are necessary for the BPS limits as the x dependence drops out. For the traces in (5.9), (5.21), as the $\mathcal{J}_m$, $\mathcal{J}_\pm$ eigenvalues are zero for states contributing to subalgebra characters, in the limit as $\delta \to 0$, then $u \to u\alpha$, $\bar s \to \bar s/\alpha$ introduces a factor of α^{-2J_3} in the trace. For $\alpha=-1$ this is equivalent to a sign change in the original variable x before the $\delta \to 0$ limit is taken.

so that $f_\pm^{U(1)\otimes U(1)}(u,y) = uy^{\pm\frac{1}{2}}$. We also have that,

$$Y_{\text{sugra}}^{U(1)\otimes U(1)}(u,y) = \sum_{r=1}^{\infty} u^{2r} = \frac{u^2}{1-u^2}\,. \tag{7.7}$$

Thus, from (7.3),

$$Z_{\text{free}}^{U(1)\otimes U(1)}(u,y) = Z_{\text{sugra}}^{U(1)\otimes U(1)}(u,y) = \prod_{j\geq 1} \frac{1}{1-u^{2j}}\,. \tag{7.8}$$

Thus, expanding over the characters in (7.6),

$$Z_{\text{free}}^{U(1)\otimes U(1)}(u,y) = Z_{\text{sugra}}^{U(1)\otimes U(1)}(u,y) = 1 + \sum_{r=1}^{\infty} N_r^{(3,B,2)} \chi_{(r;0;r,r,0)}^{(3,B,2)}(u,y)\,, \tag{7.9}$$

where

$$N_r^{(3,B,2)} = p(r) = 1,2,3,5,\ldots, \quad r = 1,2,3,4,\ldots\,, \tag{7.10}$$

where $p(r)$ is the usual partition number for r. The agreement (7.8) is expected as $(3,B,2)$ operators are protected due to long multiplet decomposition rules discussed in section 4.

The $U(1)\otimes SO(4)$ Sector

Due to (4.33), we expect disagreement between counting of $(3,B,1)$ operators in the free field and supergravity limits and thus we split the discussion here. Corresponding to (5.5) for $N=3$, $m=1$, we have that,

$$\begin{aligned}
\chi_{(r;0;r,q,p)}^{(3,B,1)}(u,u_+,u_-) &= \chi_{(r;q,p)}^{U(1)\otimes SO(4)}(u,u_+,u_-) = u^{2r}\chi_{q+p}(u_+)\chi_{q-p}(u_-)\,,\\
\chi_{(r;0;r,r,p)}^{(3,B,2)}(u,u_+,u_-) &= \chi_{(r;r,p)}^{U(1)\otimes SO(4)}(u,u_+,u_-) = u^{2r}\chi_{r+p}(u_+)\chi_{r-p}(u_-)\,,\\
\chi_{(r;0;r,r,\pm r)}^{(3,B,\pm)}(u,u_+,u_-) &= \chi_{(r;r,\pm r)}^{U(1)\otimes SO(4)}(u,u_+,u_-) = u^{2r}\chi_{2r}(u_\pm)\,,
\end{aligned} \tag{7.11}$$

in terms of $SU(2)$ characters in (4.5), using $SO(4)\simeq SU(2)\otimes SU(2)$, so that, for $SO(4)$ characters,

$$\chi_{(q,p)}^{(2)}(\bar{y}_1,\bar{y}_2) = \chi_{q+p}(u_+)\chi_{q-p}(u_-)\,, \qquad \bar{y}_1 = u_+u_-\,, \quad \bar{y}_2 = u_+/u_-\,. \tag{7.12}$$

We have $f_\pm^{U(1)\otimes SO(4)}(u) = u\chi_1(u_\pm)$, where $\chi_1(u_\pm) = u_\pm + {u_\pm}^{-1}$, so that, from (7.3),

$$Z_{\text{free}}^{U(1)\otimes SO(4)}(u,u_+,u_-) = \prod_{j=1}^{\infty} \frac{1}{1-u^{2j}\chi_1({u_+}^j)\chi_1({u_-}^j)}\,. \tag{7.13}$$

The numbers of multi-trace $(3,B,1)$ BPS operators are then determined by expansions over the characters (7.11), using (7.10),

$$\begin{aligned} & Z_{\rm free}^{U(1)\otimes SO(4)}(u,u_+,u_-) \\ & = 1+\sum_{r=1}^{\infty} p(r)\chi^{(3,B,2)}_{(r;0;r,r,0)}(u,u_+,u_-) + \sum_{r,q,|p|\geq 0}^{\infty} N^{(3,B,1)}_{{\rm free},(r,q,p)}\,\chi^{(3,B,1)}_{(r;0;r,q,p)}(u,u_+,u_-)\,, \end{aligned} \tag{7.14}$$

for which formulae are obtained in appendix B. We may determine for the first few cases, for $r=0,1,2,\ldots$,

$$N^{(3,B,1)}_{{\rm free},(r+1,r,\pm 1)} = \sum_{j=0}^{r} p(j) - p(r{+}1) = 0,1,2,5,8\ldots\,, \tag{7.15}$$

and

$$N^{(3,B,1)}_{{\rm free},(r+2,r,0)} = 2\sum_{j=1}^{r}\sum_{k=0}^{j} p(k) - \sum_{j=0}^{r+1} p(j) + p(r{+}2) = 2,5,12,23,44,\ldots\,, \tag{7.16}$$

along with,

$$N^{(3,B,1)}_{{\rm free},(r+4,r+2,\pm 2)} = 2\sum_{j=1}^{r}\sum_{k=0}^{[\frac{j}{2}]} p(\tfrac{1}{2}(1{-}(-1)^j)+2k) + \sum_{j=0}^{r+2} p(j) - p(r{+}3) = 3,6,15,26,49,\ldots\,. \tag{7.17}$$

Note that generally, from results in appendix B, $N^{(3,B,1)}_{{\rm free},(r,q,p)}$ is a potentially non-zero integer only for $N^{(3,B,1)}_{{\rm free},(r,r-s-t,s-t)}$, $r\in\mathbb{N}$, $s,t=0,\ldots,[r/2]$, $s\,t\neq 0$.

(From (6.1), $(r,r{-}s{-}t,t{-}s)\to[r{-}2s,s{+}t,r{-}2t]$, in terms of $SU(4)$ Dynkin labels.)

For the supergravity limit we may determine,

$$Y_{\rm sugra}^{U(1)\otimes SO(4)}(u,u_+,u_-) = \sum_{r=1}^{\infty}\chi^{(3,B,2)}_{(r;0;r,r,0)}(u,u_+,u_-) = \frac{1-u^4}{\prod_{\varepsilon,\eta=\pm1}(1-u^2{u_+}^{\varepsilon}{u_-}^{\eta})} - 1\,, \tag{7.18}$$

so that, from (7.3),

$$Z_{\rm sugra}^{U(1)\otimes SO(4)}(u,u_+,u_-) = \prod_{j=1}^{\infty}\frac{1}{\prod_{k,l=0}^{j}(1-u^{2j}{u_+}^{2k-j}{u_-}^{2l-j})}\,. \tag{7.19}$$

Again we may expand,

$$\begin{aligned} & Z_{\rm sugra}^{U(1)\otimes SO(4)}(u,u_+,u_-) \\ & = 1+\sum_{r=1}^{\infty} p(r)\chi^{(3,B,2)}_{(r;0;r,r,0)}(u,u_+,u_-) + \sum_{r,q,|p|\geq 0}^{\infty} N^{(3,B,1)}_{{\rm sugra},(r,q,p)}\,\chi^{(3,B,1)}_{(r;0;r,q,p)}(u,u_+,u_-)\,, \end{aligned} \tag{7.20}$$

to find for the first few cases, using results from appendix B, for $r = 0, 1, 2, \ldots$,

$$N^{(3,B,1)}_{\text{sugra},(r+1,r,\pm1)} = N^{(3,B,1)}_{\text{free},(r+1,r,\pm1)} \tag{7.21}$$

while,

$$N^{(3,B,1)}_{\text{sugra},(r+2,r,0)} = \sum_{j=0}^{r}\sum_{k=0}^{j} p(k) - \sum_{j=0}^{r+1} p(j) + p(r+2) = 1, 2, 5, 9, 18, \ldots, \tag{7.22}$$

along with,

$$N^{(3,B,1)}_{\text{sugra},(r+4,r+2,\pm2)} = \sum_{j=1}^{r}\sum_{k=0}^{[\frac{j}{2}]} p(\tfrac{1}{2}(1-(-1)^j) + 2k) + \sum_{j=0}^{r+2} p(j) - p(r+3) = 2, 4, 10, 17, 32, \ldots. \tag{7.23}$$

As emphasised, the matching (7.21) is expected from (4.33) along with the free field restrictions implied for general r, q, p in $N^{(3,B,1)}_{\text{free},(r,q,p)}$ mentioned above.

The first few numbers of operators may be easily obtained by performing series expansions to low orders in u and using the orthogonality relation for $SU(2)$ characters in appendix B and are listed in the following table.

Table 3

BPS primary operators		
Δ	Supergravity limit	Remaining operators
1	$\mathcal{R}_{(1,1,0)}$	
2	$2\mathcal{R}_{(2,2,0)}$, $\mathcal{R}_{(2,0,0)}$	$\mathcal{R}_{(2,0,0)}$
3	$3\mathcal{R}_{(3,3,0)}$, $\mathcal{R}_{(3,2,\pm1)}$, $2\mathcal{R}_{(3,1,0)}$	$3\mathcal{R}_{(3,1,0)}$
4	$5\,\mathcal{R}_{(4,4,0)}$, $2\,\mathcal{R}_{(4,3,\pm1)}$, $2\,\mathcal{R}_{(4,2,\pm2)}$, $5\,\mathcal{R}_{(4,2,0)}$ $\mathcal{R}_{(4,1,\pm1)}$, $3\mathcal{R}_{(4,0,0)}$	$\mathcal{R}_{(4,2,\pm2)}$, $7\mathcal{R}_{(4,2,0)}$ $3\mathcal{R}_{(4,1,\pm1)}$, $6\mathcal{R}_{(4,0,0)}$

BPS primary operators with conformal dimensions Δ belonging to $SO(6)$ representations $\mathcal{R}_{(r,q,p)}$, as obtained from expansion of partition functions. (For the free field case, the extra operators appear in the rightmost column.)

Note that more generally, as may be easily argued from results in appendix B, $N^{(3,B,1)}_{\text{sugra},(r,q,p)}$ is potentially non-vanishing only for $N^{(3,B,1)}_{\text{sugra},(r,r-s-t,s-t)}$, $r \in \mathbb{N}$, $s, t = 0, \ldots, [r/2]$, $s\,t \neq 0$, consistent with the free field theory result.

Semi-short Cases

The $U(1) \otimes SU(1,1)$ Sectors

In these sectors, where the relevant limits in characters are given by (5.21), for $N = 3$, fermion contributions become important in multi-particle partition functions.

The surviving characters, for the $(N, A, +)$ limit, are given by,

$$\begin{aligned}
&\chi^{(3,B,-)}_{(\frac{1}{2};0;\frac{1}{2},\frac{1}{2},-\frac{1}{2})}(u,\bar{s}) = \frac{u^{\frac{1}{2}}\,\bar{s}^{\frac{3}{2}}}{1-\bar{s}^2}\,, \qquad &&\chi^{(3,B,+)}_{(r;0;r,r,r)}(u,\bar{s}) = \frac{u^r\,\bar{s}^r}{1-\bar{s}^2}\,, \\
&\chi^{(3,B,2)}_{(r;0;r,r,r-1)}(u,\bar{s}) = \frac{u^r\,\bar{s}^{r+1}}{1-\bar{s}^2}\,, \qquad &&\chi^{(3,B,1)}_{(r;0;r,r-1,r-1)}(u,\bar{s}) = \frac{u^r\,\bar{s}^{r+2}}{1-\bar{s}^2}\,, \\
&\chi^{(3,\text{cons.})}_{(j+1;j;0,0,0)}(u,\bar{s}) = \frac{u\,\bar{s}^{2j+4}}{1-\bar{s}^2}\,, \qquad &&\chi^{(3,A,+)}_{(r+j+1;j;r,r,r)}(u,\bar{s}) = \frac{u^{r+1}\,\bar{s}^{r+2j+4}}{1-\bar{s}^2}\,.
\end{aligned} \tag{7.24}$$

Thus, due to $f_+^{U(1)\otimes SU(1,1)}(u,\bar{s}) = (u\bar{s})^{\frac{1}{2}}/(1-\bar{s}^2)$, $f_-^{U(1)\otimes SU(1,1)}(u,\bar{s}) = u^{\frac{1}{2}}\,\bar{s}^{\frac{3}{2}}/(1-\bar{s}^2)$, from (7.3),

$$Z_{\text{free}}^{U(1)\otimes SU(1,1)}(u,\bar{s}) = \prod_{j=1}^{\infty} \frac{1}{1+(-1)^j u^j\,\bar{s}^{2j}(1-\bar{s}^{2j})^{-2}}\,. \tag{7.25}$$

Expanding in u we find,

$$\begin{aligned}
Z_{\text{free}}^{U(1)\otimes SU(1,1)}(u,\bar{s}) &= 1 + \frac{u\,s^2}{(1-s^2)^2} + \frac{4u^2\,s^6}{(1-s^2)^2(1-s^4)^2} + O(u^3,s^6) \\
&= 1 + \chi^{(3,B,2)}_{(1;0;1,1,0)}(u,\bar{s}) + \sum_{j=0}^{\infty} \chi^{(3,\text{cons.})}_{(j+1;j;0,0,0)}(u,\bar{s}) \\
&\quad + \sum_{r=1}^{\infty} \sum_{\substack{j=\frac{r}{2}-1 \\ j\geq 0}}^{\infty} N^{(3,A,+)}_{\text{free},(r,j)}\,\chi^{(3,A,+)}_{(r+j+1;j;r,r,r)}(u,\bar{s})\,.
\end{aligned} \tag{7.26}$$

It appears non-trivial to determine $N^{(3,A,+)}_{\text{free},(r,j)}$ generally, however we may easily determine,

$$\begin{aligned}
&N^{(3,A,+)}_{\text{free},(1,j)} = 2[\tfrac{1}{2}j+1][\tfrac{1}{2}j+2] = 4,4,12,12,\ldots, \qquad j = \tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\tfrac{7}{2},\ldots, \\
&N^{(3,A,+)}_{\text{free},(r,\frac{r}{2}-1)} = 1\,, \quad r = 2,3,4,\ldots, \qquad N^{(3,A,+)}_{\text{free},(r,\frac{r}{2}-1+n)} > 1\,, \quad n = 1,2,3,\ldots\,.
\end{aligned} \tag{7.27}$$

In this limit, the supergravity single particle partition function reduces to,

$$Y_{\text{sugra}}^{U(1)\otimes SU(1,1)}(u,\bar{s}) = \chi^{(3,B,2)}_{(1;0;1,1,0)}(u,\bar{s}) = \frac{u\,\bar{s}^2}{1-\bar{s}^2}\,, \tag{7.28}$$

so that, from (7.3),

$$Z_{\text{sugra}}^{U(1)\otimes SU(1,1)}(u,\bar{s}) = \prod_{j=1}^{\infty}(1+u\,\bar{s}^{2j})\,, \tag{7.29}$$

which is a reflection of $Y_{\text{sugra}}^{U(1)\otimes SU(1,1)}(u,\bar{s})$ receiving purely fermionic operator contributions (it is odd under $(u,\bar{s}) \to -(u,\bar{s})$).

Using the identities,

$$\prod_{j=1}^{\infty}(1+zq^j)=\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}n(n+1)}z^n}{\prod_{j=1}^{n}(1-q^j)}\,,\qquad \prod_{j=2}^{n}\frac{1}{1-q^j}=\sum_{m=0}^{\infty}P_n(m)q^m\,, \tag{7.30}$$

where $P_n(m)$ is the number of partitions of m into no more that n parts, each part ≥ 2, we may write,

$$Z_{\rm sugra}^{U(1)\otimes SU(1,1)}(u,\bar{s})=1+\chi^{(3,B,2)}_{(1;0;1,1,0)}(u,\bar{s})+\sum_{r=1}^{\infty}\sum_{j=\frac{1}{2}r^2+r-1}^{\infty}N^{(3,A,+)}_{{\rm sugra},(r,j)}\,\chi^{(3,A,+)}_{(r+j+1;j;r,r,r)}(u,\bar{s})\,, \tag{7.31}$$

where,

$$N^{(3,A,+)}_{{\rm sugra},(r,j)}=P_{r+1}(j+1-\tfrac{1}{2}r^2-r)\,. \tag{7.32}$$

Using the foregoing we may find relatively simple formulae for at least the first few cases,

$$\begin{aligned}
N^{(3,A,+)}_{{\rm sugra},(1,j)}&=\begin{cases}1\,, & j-\frac{1}{2}=0 \bmod 2,\\ 0\,, & j-\frac{3}{2}=0 \bmod 2,\end{cases}\qquad j=\tfrac{1}{2},\tfrac{3}{2},\ldots,\\
N^{(3,A,+)}_{{\rm sugra},(2,j)}&=1,0,1,1,\ldots=\begin{cases}[\frac{1}{6}j-\frac{1}{2}]\,, & j=4 \bmod 6,\\ [\frac{1}{6}j+\frac{1}{2}]\,, & \text{otherwise},\end{cases}\qquad j=3,4,5,6,\ldots,\\
N^{(3,A,+)}_{{\rm sugra},(r,\frac{1}{2}r^2+r-1)}&=1\,.
\end{aligned} \tag{7.33}$$

Assuming that the partition functions in the supergravity limit receive contributions from only protected operators then a number of observations from (7.26), (7.27), (7.31) and (7.33) are evident.

First, there are no $\chi^{(3,B,1)}_{(r;0;r,r-1,r-1)}(u,\bar{s})$ contributions and one contribution from $\chi^{(3,B,2)}_{(r;0;r,r,r-1)}(u,\bar{s})$, for $r=1$, both consistent with previous analysis for BPS cases. Second, the conserved current contributions in (7.26) disappear from the spectrum in the strong coupling limit, as happens for $\mathcal{N}=4$ super Yang Mills. Third, there is one scalar $(3,A,+)$ semishort primary operator in the $SO(6)$ representation $\mathcal{R}_{(2,2,2)}$, and its conformal descendants, contributing in the free field theory limit, due to $N^{(3,A,+)}_{{\rm free},(2,0)}=1$ in (7.26), and no such contribution in the supergravity limit, from (7.31). In the interacting theory, these operators, from the long multiplet decomposition formulae (4.35), must then pair up with a $(3,B,1)$ BPS primary operator in the $SO(6)$ representation $\mathcal{R}_{(4,2,2)}$, and its descendants. This is consistent with Table 3 as there is precisely one such remaining $(3,B,1)$ primary operator. (Similarly, the scalar conserved current, and its descendants, contributing to (7.26) pairs with the single $(3,B,1)$ BPS primary operator in the $SO(6)$ representation $\mathcal{R}_{(2,0,0)}$, and its descendants, listed in Table 3.)

Note that the analysis for $(3,A,-)$ semishort operators is very similar and gives the same result for counting of BPS and conserved current multiplets along with,

$$N^{(3,A,-)}_{\text{free},(r,j)} = N^{(3,A,+)}_{\text{free},(r,j)}\,, \qquad N^{(3,A,-)}_{\text{sugra},(r,j)} = N^{(3,A,+)}_{\text{sugra},(r,j)}\,. \tag{7.34}$$

In particular, again there is one scalar $(3,A,-)$ primary semishort operator contributing in the free field theory limit, absent from the supergravity spectrum, that pairs up with a $(3,B,1)$ BPS primary operator in the $\mathcal{R}_{(4,2,-2)}$ $SO(6)$ representation, consistent with Table 3.

The $U(1)\otimes Osp(2|2)$ Sector

In the limit (5.9), for $N=3$, $m=2$, the non-vanishing characters are, defining $Q(\bar{s},y)=(1+\bar{s}y)(1+\bar{s}y^{-1})/(1-\bar{s}^2)$,

$$\begin{gathered}
\chi^{(3,B,\pm)}_{(r;0;r,r,\pm r)}(u,\bar{s},y) = \frac{u^r\bar{s}^r}{1-\bar{s}^2}y^{\pm r}(1+\bar{s}\,y^{\mp 1})\,, \qquad \chi^{(3,B,2)}_{(r;0;r,r,q)}(u,\bar{s},y) = (u\,\bar{s})^r\,y^q\,Q(\bar{s},y)\,,\\
\chi^{(3,B,1)}_{(r;0;r,r-1,q)}(u,\bar{s},y) = u^r\,\bar{s}^{r+1}\,y^q\,Q(\bar{s},y)\,, \qquad \chi^{(3,\text{cons.})}_{(j+1;j;0,0,0)}(u,\bar{s},y) = u\,\bar{s}^{2j+3}Q(\bar{s},y)\,,\\
\chi^{(3,A,\pm)}_{(r+j+1;j;r,r,\pm r)}(u,\bar{s},y) = u^{r+1}\,\bar{s}^{r+2j+3}\,y^{\pm r}\,Q(\bar{s},y)\,,\\
\chi^{(3,A,2)}_{(r+j+1;j;r,r,q)}(u,\bar{s},y) = u^{r+1}\,\bar{s}^{r+2j+3}\,y^{q}\,Q(\bar{s},y)\,.
\end{gathered} \tag{7.35}$$

Using that $f_\pm^{U(1)\otimes Osp(2|2)}(u,\bar{s},y)=(u\,\bar{s})^{\frac12}(y^{\pm\frac12}+\bar{s}\,y^{\mp\frac12})/(1-\bar{s}^2)$ we have,

$$Z^{U(1)\otimes Osp(2|2)}_{\text{free}}(u,\bar{s},y) = \prod_{j=1}^{\infty}\frac{1}{1-u^j\,\bar{s}^j(1-(-1)^j\bar{s}^jy^j)(1-(-1)^j\bar{s}^jy^{-j})(1-\bar{s}^{2j})^{-2}}\,. \tag{7.36}$$

Using the previous results for counting of BPS operators, we may determine,

$$W^{(3,B,2)}_{\text{free}}(u,\bar{s},y) = \sum_{r=1}^{\infty}p(r)\chi^{(3,B,2)}_{(r;0;r,r,0)}(u,\bar{s},y) = \bigg(\prod_{k=1}^{\infty}\frac{1}{1-(u\,\bar{s})^j}-1\bigg)Q(\bar{s},y)\,, \tag{7.37}$$

for contributions from $(3,B,2)$ multiplets, using (7.10), and

$$\begin{aligned}
&W^{(3,B,1)}_{\text{free}}(u,\bar{s},y)\\
&= \sum_{r=1}^{\infty}\bigg(N^{(3,B,1)}_{\text{free},(r,r-1,1)}\chi^{(3,B,1)}_{(r;0;r,r-1,1)}(u,\bar{s},y) + N^{(3,B,1)}_{\text{free},(r,r-1,-1)}\chi^{(3,B,1)}_{(r;0;r,r-1,-1)}(u,\bar{s},y)\bigg)\\
&= \bar{s}(y+y^{-1})\bigg(\frac{2u\bar{s}-1}{1-u\bar{s}}\prod_{k=1}^{\infty}\frac{1}{1-(u\,\bar{s})^j}+1\bigg)Q(\bar{s},y)\,,
\end{aligned} \tag{7.38}$$

for contributions of $(3,B,1)$ multiplets, using the form of the corresponding characters in (7.35), along with (7.15). Similarly, using the free field results for the $U(1)\otimes SU(1,1)$ sector in (7.26), we may determine,

$$W_{\rm free}^{(3,{\rm cons.})}(u,\bar s,y)=\sum_{j=0}^{\infty}\chi^{(3,{\rm cons.})}_{(j+1;j;0,0,0)}(u,\bar s,y)=\frac{u\,\bar s^3}{1-\bar s^2}Q(\bar s,y)\,, \tag{7.39}$$

for conserved current multiplet contributions and,

$$\begin{aligned}&W_{\rm free}^{(3,A,\pm)}(u,\bar s,y)\\&=\sum_{r=1}^{\infty}\sum_{\substack{j=\frac{r}{2}-1\\ j\geq 0}}^{\infty}N^{(3,A,\pm)}_{{\rm free},(r,j)}\,\chi^{(3,A,\pm)}_{(r+j+1;j;r,r,\pm r)}(u,\bar s,y)\\&=y^{\mp1}\bar s^{-1}(1+\bar s\,y)(1+\bar s\,y^{-1})\Bigg(\prod_{k=1}^{\infty}\frac{1}{1+(-1)^j(uy^{\pm1}\bar s^2)^j(1-\bar s^{2j})^{-2}}-\frac{u\,y^{\pm1}\bar s}{(1-\bar s^2)^2}-1\Bigg)\,,\end{aligned} \tag{7.40}$$

for $(3,A,\pm)$ semishort multiplets, evident by using (7.26), (7.34), and the form of the corresponding characters in (7.35). We may then determine numbers $N^{(3,A,2)}_{{\rm free},(r,j,q)}$ of $(3,A,2)$ semishort operators from,

$$\begin{aligned}Z_{\rm free}^{U(1)\otimes Osp(2|2)}(u,\bar s,y)=1&+W_{\rm free}^{(3,B,2)}(u,\bar s,y)+W_{\rm free}^{(3,B,1)}(u,\bar s,y)+W_{\rm free}^{(3,{\rm cons.})}(u,\bar s,y)\\&+W_{\rm free}^{(3,A,+)}(u,\bar s,y)+W_{\rm free}^{(3,A,-)}(u,\bar s,y)\\&+\sum_{\substack{r\geq1,2j\geq0\\0\leq|q|<r}}N^{(3,A,2)}_{{\rm free},(r,j,q)}\,\chi^{(3,A,2)}_{(r+j+1;j;r,r,q)}(u,\bar s,y)\,.\end{aligned} \tag{7.41}$$

Finding general formulae for $N^{(3,A,2)}_{{\rm free},(r,j,q)}$ is nontrivial, however series expansion of (7.41), using Mathematica, suggests the following results for particular cases, for $r=0,1,2,\ldots$,

$$N^{(3,A,2)}_{{\rm free},(r+1,0,0)}=2\sum_{j=0}^{r}\sum_{k=0}^{j}p(k)+\sum_{j=0}^{r+1}p(j)=4,10,21,40,71,\ldots\,, \tag{7.42}$$

along with,

$$N^{(3,A,2)}_{{\rm free},(r+3,0,\pm2)}=2\sum_{j=0}^{r+1}\sum_{k=0}^{[\frac{j}{2}]}p(\tfrac12(1-(-1)^j)+2k)-\sum_{j=0}^{r+3}p(j)+p(r+4)=2,5,10,19,33,\ldots\,. \tag{7.43}$$

For the $U(1)\otimes Osp(2|2)$ sector, the supergravity single particle partition function reduces to,

$$Y_{\rm sugra}^{U(1)\otimes Osp(2|2)}(u,\bar s,y)=\sum_{r=1}^{\infty}\chi^{(3,B,2)}_{(r;0;r,r,0)}(u,\bar s,y)=\frac{u\,\bar s(1+\bar s\,y)(1+\bar s\,y^{-1})}{(1-\bar s^2)(1-u\,\bar s)}\,, \tag{7.44}$$

so that, from (7.3),

$$Z_{\rm sugra}^{U(1)\otimes Osp(2|2)}(u,\bar s,y) = \prod_{j,k=1}^{\infty} \frac{(1+u^j\,\bar s^{j+2k-1}y)(1+u^j\,\bar s^{j+2k-1}y^{-1})}{(1-u^j\,\bar s^{j+2k-2})(1-u^j\,\bar s^{j+2k})}\,. \tag{7.45}$$

This time, we may determine,

$$\begin{aligned} &W_{\rm sugra}^{(3,B,2)}(u,\bar s,y) = W_{\rm free}^{(3,B,2)}(u,\bar s,y)\,,\\ &W_{\rm sugra}^{(3,B,1)}(u,\bar s,y) = W_{\rm free}^{(3,B,1)}(u,\bar s,y)\,, \qquad W_{\rm sugra}^{(3,{\rm cons.})}(u,\bar s,y) = 0\,, \end{aligned} \tag{7.46}$$

due to the relevant sector of BPS operators remaining protected and the conserved current multiplet operators disappearing from the spectrum, as shown previously. Similarly, using the supergravity limit results for the $U(1)\otimes SU(1,1)$ sector in (7.31), we may determine,

$$\begin{aligned} &W_{\rm free}^{(3,A,\pm)}(u,\bar s,y)\\ &= \sum_{r=1}^{\infty}\sum_{\substack{j=\frac{r}{2}-1\\ j\geq 0}}^{\infty} N^{(3,A,\pm)}_{{\rm sugra},(r,j)}\,\chi^{(3,A,\pm)}_{(r+j+1;j;r,r,\pm r)}(u,\bar s,y)\\ &= y^{\mp1}\bar s^{-1}(1+\bar s\,y)(1+\bar s\,y^{-1})\bigg(\prod_{k=1}^{\infty}(1+u\,y^{\pm1}\,\bar s^{2j}) - \frac{u\,y^{\pm1}\bar s}{(1-\bar s^2)^2} - 1\bigg)\,, \end{aligned} \tag{7.47}$$

for $(3,A,\pm)$ semishort multiplets, evident by using also (7.34), and the form of the corresponding characters in (7.35). We may then determine numbers $N^{(3,A,2)}_{{\rm sugra},(r,j,q)}$ of $(3,A,2)$ semishort operators from,

$$\begin{aligned} Z_{\rm sugra}^{U(1)\otimes Osp(2|2)}(u,\bar s,y) = 1 &+ W_{\rm sugra}^{(3,B,2)}(u,\bar s,y) + W_{\rm sugra}^{(3,B,1)}(u,\bar s,y)\\ &+ W_{\rm sugra}^{(3,A,+)}(u,\bar s,y) + W_{\rm sugra}^{(3,A,-)}(u,\bar s,y)\\ &+ \sum_{\substack{r\geq 1,2j\geq 0\\ 0\leq |q|<r}} N^{(3,A,2)}_{{\rm sugra},(r,j,q)}\,\chi^{(3,A,2)}_{(r+j+1;j;r,r,q)}(u,\bar s,y)\,. \end{aligned} \tag{7.48}$$

Again, general formulae for $N^{(3,A,2)}_{{\rm sugra},(r,j,q)}$ seem nontrivial to obtain, however using Mathematica suggests, for particular cases, for $r=0,1,2,\ldots$,

$$N^{(3,A,2)}_{{\rm sugra},(1,j,0)} = 1\,, \qquad j=0,1,2,\ldots\,, \tag{7.49}$$

and

$$N^{(3,A,2)}_{{\rm sugra},(r+1,0,0)} = \sum_{j=0}^{r}\sum_{k=0}^{j} p(k) = 1,3,7,14,26,\ldots\,, \tag{7.50}$$

along with,

$$N^{(3,A,2)}_{\text{sugra},(r+3,0,\pm 2)} = \sum_{j=0}^{r+1}\sum_{k=0}^{[\frac{j}{2}]} p(\tfrac{1}{2}(1-(-1)^j)+2k) - \sum_{j=0}^{r+3} p(j) + p(r+4) = 0,0,1,2,5,\dots\,. \quad (7.51)$$

A consistency check is provided by (4.34) which implies that the unprotected scalar $(3,A,2)$ semishort primary operators in $SO(6)$ representation $\mathcal{R}_{(r,r,0)}$, respectively $\mathcal{R}_{(r,r,\pm 2)}$, and their conformal descendants, counted above, should pair with unprotected $(3,B,1)$ BPS primary operators in the $SO(6)$ representation $\mathcal{R}_{(r+2,2,0)}$, respectively $\mathcal{R}_{(r+2,r,\pm 2)}$, and their descendants, counted previously. Using (7.16), (7.22), (7.42) and (7.50), along with (7.17), (7.23), (7.42) and (7.51), we find,

$$\begin{aligned} N^{(3,B,1)}_{\text{free},(r+2,r,0)} - N^{(3,B,1)}_{\text{sugra},(r+2,r,0)} &= N^{(3,A,2)}_{\text{free},(r,0,0)} - N^{(3,A,2)}_{\text{sugra},(r,0,0)} = \sum_{j=0}^{r}\sum_{k=0}^{j} p(k)\,, \\ N^{(3,B,1)}_{\text{free},(r+2,r,\pm 2)} - N^{(3,B,1)}_{\text{sugra},(r+2,r,\pm 2)} &= N^{(3,A,2)}_{\text{free},(r,0,\pm 2)} - N^{(3,A,2)}_{\text{sugra},(r,0,\pm 2)} \\ &= \sum_{j=0}^{r-2}\sum_{k=0}^{[\frac{j}{2}]} p(\tfrac{1}{2}(1-(-1)^j)+2k)\,, \end{aligned} \quad (7.52)$$

expressing perfect agreement with this expectation.

The $U(1)\otimes Osp(4|2)$ Sector

This sector is nontrivial to analyse in similar terms as above, due to necessary and nontrivial expansions over $SO(4)$ characters, as done for the $U(1)\otimes SO(4)$ sector above, so here are simply given formulae for (7.1). Using (5.15), for $N=3$, $m=1$, with (7.12), we have,

$$\begin{aligned} &\chi^{(3,B,2)}_{(r;0;r,r,0)}(u,\bar{s},u_+,u_-) = \chi^{(U(1)\otimes Osp(2|2),2)}_{(r;r;r,0)}(u,\bar{s},u_+,u_-) \\ &= \frac{(u\,\bar{s})^r}{1-\bar{s}^2}\sum_{\varepsilon,\eta=\pm 1}\frac{(u_+)^{\varepsilon r}(u_-)^{\eta r}(1+\bar{s}\,u_+{}^{\varepsilon}u_-{}^{-\eta})(1+\bar{s}\,u_+{}^{-\varepsilon}u_-{}^{\eta})(1+\bar{s}\,u_+{}^{-\varepsilon}u_-{}^{-\eta})}{(1-u_+{}^{-2\varepsilon})(1-u_-{}^{-2\eta})}\,, \end{aligned} \quad (7.53)$$

so that, with (5.18),

$$\begin{aligned} &f^{U(1)\otimes Osp(4|2)}_{\pm}(u,\bar{s},u_+,u_-) = \frac{(u\,\bar{s})^{\frac{1}{2}}}{1-\bar{s}^2}\Big(\chi_1(u_\pm)+\bar{s}\chi_1(u_\mp)\Big)\,, \\ &Y^{U(1)\otimes Osp(4|2)}_{\text{sugra}}(u,\bar{s},u_+,u_-) \\ &= \frac{1}{1-\bar{s}^2}\sum_{\varepsilon=\pm 1}\frac{(1+u\,\bar{s}^2u_+{}^{2\varepsilon})(1+\bar{s}(u_+\,u_-)^{-\varepsilon})(1+\bar{s}(u_+/u_-)^{-\varepsilon})}{(1-u_+{}^{-2\varepsilon})(1-u\,\bar{s}(u_+\,u_-)^{\varepsilon})(1-u\,\bar{s}(u_+/u_-)^{\varepsilon})} - 1\,. \end{aligned} \quad (7.54)$$

(7.54) is consistent with (6.18) in the limit (5.9) and the formula in the second line can also be shown to be symmetric under exchange of u_+, u_-, as is necessary.

8. Conclusion

While much progress has been made recently in terms of determining the spectra of superconformal field theories, there are many open questions, for instance, for the new superconformal Chern Simons theories or $\mathcal{N}=4$ super Yang Mills.

Focussing on the former, while the spectra of the $\mathcal{N}=6$ superconformal Chern Simons theory at zero (effective) 't Hooft coupling and in the large n,k limits has been partially addressed here by use of character methods, it may be interesting to investigate operator counting for finite n,k.

For large n,k, the results here provide extra confirmation of expectations that the primary operators dual to Kaluza Klein modes, and multi-traces of these operators, in $[r,0,r]$ $SU(4)$ representations, conformal dimension r, are protected, as argued in another way in [2]. Also, the counting here implies that these are the only gauge invariant multi-trace primary operators in the $(3,B,2)$ superconformal representation.

For the $(3,B,1)$ representations, the only gauge invariant primary operators belong to $[r-2s,s+t,r-2t]$, $s,t=0,\ldots,[r/2]$, $s\,t\neq 0$, $SU(4)$ representations, for which generating functions for counting are given in appendix B. Furthermore, in accord with long multiplet decomposition rules, counting of $(s,t)=(1,0)$ and $(0,1)$ cases shows matching between the free field and supergravity limits, providing a consistency check of the character procedure used here and further evidence, perhaps, in favour of the duality proposed in [2].

Turning to semishort cases, conserved current multiplet operators disappear from the spectrum in the supergravity limit while the primary operators for $(3,A,\pm)$ semishort operators, belonging to $[2r,0,0]$, $[0,0,2r]$ $SU(4)$ representations, have spins $j\geq\frac{1}{2}r-1$ in the free field limit, while for the supergravity limit, $j\geq\frac{1}{2}r^2+r-1$, a reflection of the very simple partition function obtained in (7.45).

While only partial counting is obtained here for semishort operators, the numbers of primary operators obtained are consistent with the following formula, implied by (4.34), (4.35), (4.36),

$$N^{\mathcal{M}}_{\text{prot},(r,q)}=N^{\mathcal{M}}_{\text{free},(r,q)}-N^{(3,B,1)}_{\text{free},(r+2,r,q)}+N^{(3,B,1)}_{\text{prot},(r+2,r,q)}\,, \tag{8.1}$$

where $N^{\mathcal{M}}_{\text{prot},(r,q)}$, $N^{\mathcal{M}}_{\text{free},(r,q)}$ denote numbers of corresponding protected/free *scalar* semishort primary operators, in appropriate $\mathcal{R}_{(r,r,q)}$ $SO(6)$ representations (so that for $\mathcal{M}=(3,\text{cons.})$, then $r,q=0$, for $\mathcal{M}=(3,A,\pm)$ then $r=\pm q\neq 0$ and for $\mathcal{M}=(3,A,2)$ then $r>|q|>0$) while $N^{(3,B,1)}_{\text{prot},(r+2,r,q)}$, $N^{(3,B,1)}_{\text{free},(r+2,r,q)}$ denote numbers of protected/free primary operators in $(3,B,1)$ BPS multiplets, in $\mathcal{R}_{(r+2,r,q)}$ $SO(6)$ representations.

Note also that, for free field theory in the large n limit, r,q are further restricted

to $r = 2$, $q = \pm 2$ for the $\mathcal{M} = (3, A, \pm)$ cases and $r \in \mathbb{N}$, $q = 0, \pm 2$, $r > |q|$, for the $\mathcal{M} = (3, A, 2)$ cases.

(8.1) implies that it is possible to compute the partition function corresponding to such protected operators using only free field theory and the knowledge of which $(3, B, 1)$ BPS operators remain protected.

In particular this applies to the $\mathcal{N} = 6$ Chern Simons theory, having finite n and large levels k, in which case allowable r, q in (8.1), deriving from free field theory, may change. The generating functions,

$$\begin{aligned} F_q(z) &= \sum_{r\geq 0} N^{\mathcal{M}}_{\text{free},(r,q)} z^r \,, \\ G_q(z) &= \sum_{r\geq 0} \Big(N^{(3,B,1)}_{\text{free},(r+2,r,q)} - N^{(3,B,1)}_{\text{prot},(r+2,r,q)} \Big) z^r \,, \end{aligned} \tag{8.2}$$

may then be determined using the free field partition functions $Z^{(n),U(1)\otimes Osp(2|2)}_{\text{free}}(u, \bar{s}, y)$, $Z^{(n),U(1)\otimes SO(4)}_{\text{free}}(u, u_+, u_-)$, denoting (6.6) evaluated in the relevant $H \subset Osp(6|4)$ sector, and $Z^{(n),U(1)\otimes SO(4)}_{\text{prot}}(u, u_+, u_-)$, the partition function for protected $(3, B, 1)$ operators evaluated in the $U(1) \otimes SO(4)$ sector.[12] (Presumably the latter partition function is equivalent to the chiral ring partition function, for finite n.)

These generating functions may then be used to write, using also (4.7), (4.27), (4.28) for $N = 3$,

$$\begin{aligned} Z_q(s, x, y_1, y_2, y_3) &= \sum_{r\geq 0} N^{\mathcal{M}}_{\text{prot},(r,q)} \chi^{(3,A,1)}_{(r+1;0;r,r,q)}(s, x, y_1, y_2, y_3) \\ &= \mathfrak{W}^{(3)} \Big(H_q(s^2 y_1 y_2)\, {y_3}^q\, C^{(3,A,1)}_{(1;0;0,0,0)}(s, x, y_1, y_2, y_3) \Big) \,, \\ H_q(z) &= F_q(z) - G_q(z) \,, \end{aligned} \tag{8.3}$$

for the partition function restricted to relevant protected scalar primary operators, and their superconformal descendants.

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[12] It is evident from the form of the characters in (7.35), that,

$$F_q(z) = \frac{1}{(2\pi i)^2} \oint\oint \frac{1 - x^2}{x^3 y^q (x + y)(1 + xy)} \Big(Z^{(n),U(1)\otimes Osp(2|2)}_{\text{free}}(z/x, x, y) - 1 \Big) \mathrm{d}x \, \mathrm{d}y \,,$$

in terms of a double contour integral, where the contribution from the identity operator is subtracted. (Depending on free field constraints for finite n it may also be necessary to subtract contributions from other short multiplet representations as well.) $G_q(z)$ may, in principle, be computed in terms of a double contour integral similar to that in appendix B, where $f(u, x, y) = Z^{(n),U(1)\otimes SO(4)}_{\text{free}}(u^{\frac{1}{2}}, x, y) - Z^{U(1)\otimes SO(4)}_{\text{prot}}(u^{\frac{1}{2}}, x, y)$ in (B.1).

Such partition functions should be consistent with the large n result where, using that there are no relevant scalar $(3,\text{cons.})$, $(3,A,\pm)$ primary operators in the supergravity spectrum and that $N^{(3,A,2)}_{\text{prot},(r,q)} = N^{(3,A,2)}_{\text{sugra},(r,0,q)}$ with (7.50), (7.51), then

$$\begin{aligned} H_0(z) &= \sum_{r=1}^{\infty} N^{(3,A,2)}_{\text{sugra},(r,0,0)}\, z^r = \frac{z}{(1-z)^2}\prod_{k=1}^{\infty}\frac{1}{1-z^k}\,, \\ H_{\pm 2}(z) &= \sum_{r=3}^{\infty} N^{(3,A,2)}_{\text{sugra},(r,0,\pm 2)}\, z^r = \left(\frac{z^2}{(1-z)(1-z^2)} - \frac{1}{1-z} + \frac{1}{z}\right)\prod_{k=1}^{\infty}\frac{1}{1-z^k} - \frac{1}{z}\,, \end{aligned} \tag{8.4}$$

with other $H_q(z)$ vanishing.

Turning to $\mathcal{N}=4$ super Yang Mills, with gauge group $U(N)$, there are still many outstanding questions regarding operator spectra that perhaps may be more easily answered for $\mathcal{N}=6$ superconformal Chern Simons theories, due to the latter having fewer decoupled sectors, as demonstrated here. One concerns finite N counting for protected operators. (Addressing this question may help answer the difficult black hole entropy/microstate counting problem.)

Recently, the latter issue received some attention in [24], where, among other things, the problem of counting certain gauge invariant operators, consisting of products of single trace chiral ring operators acted on by derivatives, was considered. (These operators should remain protected and certainly enumerating them is interesting and worthwhile.)

The simpler problem of counting chiral ring operators has been addressed from the perspective of computing Hilbert series for the ring of symmetric polynomials in [25] (equivalently, the latter issue has been investigated from a perhaps more formal perspective in [26]) whereby the non-trivial part of the computation is in taking account of syzygies (or trace identities in terms of matrices). The same difficulty applies to counting operators of the sort, referred to above, considered in [24] and perhaps could be circumvented by a judicious choice of basis for the operators.

In appendix C, a more combinatorial approach is described using a natural basis for multi-trace operators involving commuting matrices, the simplest sort of chiral ring that is relevant for $\mathcal{N}=4$ super Yang Mills. The technique involves employing the Polya enumeration theorem, applied to counting graph colourings where the relevant graphs, in this case, have a particular wreath product group automorphism symmetry. (This is similar to the approach used for counting single trace operators, in the large N limit, for $\mathcal{N}=4$ SYM, [27,28], where the relevant graphs there were necklaces with cyclic group automorphism symmetry.) The result obtained by this approach (equivalent to the Hilbert series mentioned above) is naturally expressed in terms of cycle index polynomials for the symmetric permutation group. (For alternative polynomial expansions of chiral ring

partition functions, useful in the context of asymptotics, see, for example, [29].)

It may be worth trying to find other combinatorial methods of counting chiral ring operators more generally, perhaps using known extensions of the Polya enumeration theorem, taking into account symmetry in colours [30], for example. In any case, Hilbert series appear to have very interesting connections with counting graph colourings that perhaps may lead to even other extensions of the Polya enumeration theorem.

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Appendix A. $Osp(2N|4)$ Subalgebras

Corresponding to the shortening conditions (3.5), (3.8), (3.11), (3.14), with notation as in Table 1, there are subalgebras which are now discussed. The characters for these subalgebras lead to the limits considered in section 5.

Corresponding to (3.5) for (N, B, m) short multiplets we have,

$$Osp(2N|4) \supset \big(SU(2|m) \otimes SO(2N-2m)\big) \ltimes U(1)_{\mathcal{I}_m} \,. \tag{A.1}$$

The generators of $SU(2|m)$ are $M_\alpha{}^\beta$, $\mathcal{Q}_{\hat{m}\alpha}$, $\bar{\mathcal{S}}_{\hat{n}}{}^\beta$, $\hat{m}, \hat{n} = 1, \dots, m$, of section 2, along with $\mathcal{H}_m$ as in (5.1), and $T_{\hat{m}\hat{n}}$, generators of $SU(m) \subset SO(2m)$, with, in terms of the generators in (2.7),

$$T_{\hat{m}\hat{m}} = H_{\hat{m}} - \tfrac{1}{m}\sum_{\hat{n}=1}^{m} H_{\hat{n}} \,, \qquad T_{\hat{m}\hat{n}} = \begin{cases} \tfrac{i}{2}E^{+-}_{\hat{m}\hat{n}} \,, & \hat{m} < \hat{n} \\ -\tfrac{i}{2}E^{-+}_{\hat{n}\hat{m}} \,, & \hat{n} < \hat{m} \end{cases} \,, \tag{A.2}$$

so that $\sum_{\hat{m}=1}^{m} T_{\hat{m}\hat{m}} = 0$ and, for $\hat{k}, \hat{l} = 1, \dots, m$,

$$[T_{\hat{m}\hat{n}}, T_{\hat{k}\hat{l}}] = \delta_{\hat{n}\hat{k}}\, T_{\hat{m}\hat{l}} - \delta_{\hat{m}\hat{l}}\, T_{\hat{n}\hat{k}} \,. \tag{A.3}$$

$SU(2|m)$ has usual algebra with, in particular,

$$\{\mathcal{Q}_{\hat{m}\alpha}, \bar{\mathcal{S}}_{\hat{n}}{}^\beta\} = 2i(M_\alpha{}^\beta\, \delta_{\hat{m}\hat{n}} - \delta_\alpha{}^\beta\, T_{\hat{m}\hat{n}} + \delta_\alpha{}^\beta\, \delta_{\hat{m}\hat{n}}\, \mathcal{H}_m) \,, \tag{A.4}$$

and

$$[\mathcal{H}_m, \mathcal{Q}_{\hat{m}\alpha}] = (\tfrac{1}{2} - \tfrac{1}{m})\mathcal{Q}_{\hat{m}\alpha} \,, \qquad [\mathcal{H}_m, \bar{\mathcal{S}}_{\hat{n}}{}^\beta] = -(\tfrac{1}{2} - \tfrac{1}{m})\bar{\mathcal{S}}_{\hat{n}}{}^\beta \,. \tag{A.5}$$

For $m = 2$, $\mathcal{H}_m$ is evidently then a central extension.

The $SO(2N-2m)$ subgroup in (A.1) is generated by $R_{\bar{r}\,\bar{s}}$ $\bar{r}, \bar{s}, \bar{t}, \bar{u} = 2m+1, \dots, 2N$,

$$[R_{\bar{r}\bar{s}}, R_{\bar{t}\bar{u}}] = i(\delta_{\bar{r}\bar{t}}\, R_{\bar{s}\bar{u}} - \delta_{\bar{s}\bar{t}}\, R_{\bar{r}\bar{u}} - \delta_{\bar{r}\bar{u}}\, R_{\bar{s}\bar{t}} + \delta_{\bar{s}\bar{u}}\, R_{\bar{r}\bar{t}}) \,, \tag{A.6}$$

while $\mathcal{I}_m$ in (5.1) is an external automorphism with,

$$[\mathcal{I}_m, \mathcal{Q}_{\hat{m}\alpha}] = (\tfrac{1}{2} + \tfrac{1}{m})\mathcal{Q}_{\hat{m}\alpha} \,, \qquad [\mathcal{I}_m, \bar{\mathcal{S}}_{\hat{n}}{}^\beta] = -(\tfrac{1}{2} + \tfrac{1}{m})\bar{\mathcal{S}}_{\hat{n}}{}^\beta \,. \tag{A.7}$$

Corresponding to (3.5) for $n = N$ for $(N, B, +)$ multiplets and, separately, (3.8) for $(N, B, -)$ multiplets we have,

$$Osp(2N|4) \supset SU(2|N) \ltimes U(1)_{\mathcal{I}_\pm} \,. \tag{A.8}$$

The generators of $SU(2|N)$ are $M_\alpha{}^\beta$, $\mathcal{Q}_{\hat m\alpha}$, $\bar{\mathcal{S}}_{\hat n}{}^\beta$, for $\hat m,\hat n = 1,\dots,N{-}1,+$, for the $(N,B,+)$ case, and $\hat m,\hat n = 1,\dots,N{-}1,-$, for the $(N,B,-)$ case, where we define

$$\mathcal{Q}_{+\alpha} = \mathcal{Q}_{N\alpha}\,, \qquad \mathcal{Q}_{-\alpha} = \bar{\mathcal{Q}}_{N\alpha}\,, \qquad \bar{\mathcal{S}}_+{}^\alpha = \bar{\mathcal{S}}_N{}^\alpha\,, \qquad \bar{\mathcal{S}}_-{}^\alpha = \mathcal{S}_N{}^\alpha\,, \tag{A.9}$$

along with $\mathcal{H}_\pm$ as in (5.7), and $T_{\hat m\hat n}$, generators of $SU(N)$ given by, for $\hat m < N$,

$$\begin{aligned} &T_{\hat m\hat m} = H_{\hat m} - \tfrac{1}{N}(H_1 + \ldots + H_{N-1} \pm H_N)\,, \\ &T_{\pm\pm} = \pm H_N - \tfrac{1}{N}(H_1 + \ldots + H_{N-1} \pm H_N)\,, \\ &T_{\hat m\pm} = \tfrac{i}{2}E^{+\pm}_{\hat m N}\,, \quad T_{\pm\hat m} = -\tfrac{i}{2}E^{-\pm}_{\hat m N}\,, \quad T_{\hat m\hat n} = \begin{cases} \tfrac{i}{2}E^{+-}_{\hat m\hat n}\,, & \hat m < \hat n < N \\ -\tfrac{i}{2}E^{-+}_{\hat n\hat m}\,, & \hat n < \hat m < N \end{cases}\,, \end{aligned} \tag{A.10}$$

satisfying the same commutation relation (A.3) for $\hat m,\hat n,\hat k,\hat l = 1,\dots,N{-}1,\pm$.

The $\mathcal{Q}_{\hat m\alpha}$, $\bar{\mathcal{S}}_{\hat n}{}^\beta$ generators satisfy (A.4) for $\mathcal{H}_m$ replaced by $\mathcal{H}_\pm$, as appropriate, and $\hat m,\hat n = 1,\dots,N{-}1,\pm$, and with,

$$[\mathcal{H}_\pm, \mathcal{Q}_{\hat m\alpha}] = (\tfrac12 - \tfrac1N)\mathcal{Q}_{\hat m\alpha}\,, \qquad [\mathcal{H}_\pm, \bar{\mathcal{S}}_{\hat n}{}^\beta] = -(\tfrac12 - \tfrac1N)\bar{\mathcal{S}}_{\hat n}{}^\beta\,, \tag{A.11}$$

so that for R-symmetry group $SO(4)$, *i.e.* $N=2$, $\mathcal{H}_\pm$ is a central extension.

$\mathcal{I}_\pm$ in (5.7) acts and as external automorphism with,

$$[\mathcal{I}_\pm, \mathcal{Q}_{\hat m\alpha}] = (\tfrac12 + \tfrac1N)\mathcal{Q}_{\hat m\alpha}\,, \qquad [\mathcal{I}_\pm, \bar{\mathcal{S}}_{\hat n}{}^\beta] = -(\tfrac12 + \tfrac1N)\bar{\mathcal{S}}_{\hat n}{}^\beta\,. \tag{A.12}$$

The expression (5.2) may be understood as follows. By taking the limit $\delta \to 0$, only (N,B,n), $n \ge m$, and $(N,B,\pm)$ BPS multiplets have states, including the highest weight state, with zero $\mathcal{H}_m$ eigenvalues. Characters for the $\big(SU(2|m)\otimes SO(2N{-}2m)\big)\ltimes U(1)_{\mathcal{I}_m}$ subalgebra, when restricted to these representations, reduce to $U(1)_{\mathcal{I}_m}\otimes SO(2N{-}2m)$ characters, as these representations are trivial representations of the $SU(2|m)$ subalgebra, evident from section 3.

The limit in (5.8) may be understood similarly to other BPS cases whereby the characters reduce in the half BPS cases to $U(1)_{\mathcal{I}_\pm}$ characters.

Corresponding to (3.11) for (N,A,m) semi-short multiplets we have,

$$Osp(2N|4) \supset \big(SU(1|m)\otimes Osp(2N{-}2m|2)\big)\ltimes U(1)_{\mathcal{K}_m}\,. \tag{A.13}$$

The generators of $SU(1|m)$ are $\mathcal{Q}_{\hat m 2}, \bar{\mathcal{S}}_{\hat n}{}^2, T_{\hat m\hat n}$, $\hat m,\hat n = 1,\dots,m$, along with $\mathcal{J}_m$ as in (5.9) where $SU(1|m)$ has usual algebra with in particular,

$$\{\mathcal{Q}_{\hat m2}, \bar{\mathcal{S}}_{\hat n}{}^2\} = 2i(-T_{\hat m\hat n} + \delta_{\hat m\hat n}\,\mathcal{J}_m)\,, \tag{A.14}$$

with,

$$[\mathcal{J}_m, \mathcal{Q}_{\hat{m}2}] = (1 - \tfrac{1}{m})\mathcal{Q}_{\hat{m}2} \,, \qquad [\mathcal{J}_m, \bar{\mathcal{S}}_{\hat{n}}{}^2] = -(1 - \tfrac{1}{m})\bar{\mathcal{S}}_{\hat{n}}{}^2 \,. \tag{A.15}$$

For $m = 1$, $\mathcal{J}_m$ is then a central extension.

The generators of $Osp(2N{-}2m|2)$ are $Sp(2,\mathbb{R}) \simeq SU(1,1)$ generators P_{11}, K^{11} of section 2, along with, $\bar{D} = D + J_3$ in (5.9) satisfying,

$$[\bar{D}, P_{11}] = 2P_{11} \,, \quad [\bar{D}, K^{11}] = -2K^{11} \,, \quad [K^{11}, P_{11}] = 4\bar{D} \,, \tag{A.16}$$

along with $Q_{\bar{r}1}$, $S_{\bar{s}}{}^1$, $\bar{r}, \bar{s} = 2m{+}1, \ldots, 2N$, of section 2, and $SO(2N-2m)$ generators $R_{\bar{r}\bar{s}}$ as before, satisfying,

$$\begin{aligned} [\bar{D}, Q_{\bar{r}1}] &= Q_{\bar{r}1} \,, \qquad [\bar{D}, S_{\bar{s}}{}^1] = -S_{\bar{s}}{}^1 \,, \\ [K^{11}, Q_{\bar{r}1}] &= 2iS_{\bar{r}}{}^1 \,, \qquad [P_{11}, S_{\bar{s}}{}^1] = -2iQ_{\bar{s}1} \,, \end{aligned} \tag{A.17}$$

and

$$\{Q_{\bar{r}1}, Q_{\bar{s}1}\} = 2\delta_{\bar{r}\bar{s}}P_{11} \,, \qquad \{S_{\bar{r}}{}^1, S_{\bar{s}}{}^1\} = 2\delta_{\bar{r}\bar{s}}K^{11} \,, \qquad \{Q_{\bar{r}1}, S_{\bar{s}}{}^1\} = 2i\,\delta_{\bar{r}\bar{s}}\bar{D} + 2R_{\bar{r}\bar{s}} \,. \tag{A.18}$$

In (A.13), $\mathcal{K}_m$ as in (5.9), is an outer automorphism with,

$$[\mathcal{K}_m, \mathcal{Q}_{\hat{m}2}] = \tfrac{1}{m}\mathcal{Q}_{\hat{m}2} \,, \quad [\mathcal{K}_m, \bar{\mathcal{S}}_{\hat{n}}{}^2] = -\tfrac{1}{m}\bar{\mathcal{S}}_{\hat{n}}{}^2 \,, \quad [\mathcal{K}_m, Q_{\bar{r}1}] = [\mathcal{K}_m, S_{\bar{s}}{}^1] = 0 \,. \tag{A.19}$$

Corresponding to (3.11) for $n = N$ for $(N, A, +)$ multiplets, and separately, (3.14) for $(N, A, -)$ semi-short multiplets we have,

$$Osp(2N|4) \supset \big(SU(1|N) \otimes SU(1,1)\big) \ltimes U(1)_{\mathcal{K}_\pm} \,. \tag{A.20}$$

The generators of $SU(1|N)$ are $\mathcal{Q}_{\hat{m}2}, \bar{\mathcal{S}}_{\hat{n}}{}^2, T_{\hat{m}\hat{n}}$, for $\hat{m}, \hat{n} = 1, \ldots, N{-}1, +$, for the $(N, A, +)$ case, and $\hat{m}, \hat{n} = 1, \ldots, N{-}1, -$, for the $(N, A, -)$ case, with the definitions (A.9) for $\alpha = 2$, with $T_{\hat{m}\hat{n}}$ as in (A.10) and $\mathcal{J}_\pm$ as in (5.21).

The supercharges satisfy (A.14) with $\mathcal{J}_m$ replaced by $\mathcal{J}_\pm$, as appropriate, and $\hat{m}, \hat{n} = 1, \ldots, N{-}1, \pm$, with,

$$[\mathcal{J}_\pm, \mathcal{Q}_{\hat{m}2}] = (1 - \tfrac{1}{N})\mathcal{Q}_{\hat{m}2} \,, \qquad [\mathcal{J}_\pm, \bar{\mathcal{S}}_{\hat{n}}{}^2] = -(1 - \tfrac{1}{N})\bar{\mathcal{S}}_{\hat{n}}{}^2 \,, \tag{A.21}$$

so that, for R symmetry group $SO(2)$, *i.e.* $N = 1$, $\mathcal{J}_\pm$ is a central extension. The $SU(1,1)$ generators are P_{11}, K^{11} and $\bar{D}$, as above, satisfying (A.16). Also, $\mathcal{K}_\pm$ as given in (5.21) is an outer automorphism with,

$$[\mathcal{K}_\pm, \mathcal{Q}_{\hat{m}2}] = \tfrac{1}{N}\mathcal{Q}_{\hat{m}2} \,, \quad [\mathcal{K}_\pm, \bar{\mathcal{S}}_{\hat{n}}{}^2] = -\tfrac{1}{N}\bar{\mathcal{S}}_{\hat{n}}{}^2 \,. \tag{A.22}$$

(5.10) can be understood similarly to BPS limits. Again by taking the limit $\delta \to 0$, only those states in relevant semishort multiplets and various BPS multiplets with zero $\mathcal{J}_m$ eigenvalue contribute to corresponding characters. Note, however, that for semishort multiplets, the states do not include highest weight states, but some superconformal descendants. Such characters, when restricted to the $\big(SU(1|m) \otimes Osp(2N-2m|2)\big) \ltimes U(1)_{\mathcal{K}_m}$ subalgebra, reduce to $U(1)_{\mathcal{K}_m} \otimes Osp(2N-2m|2)$ characters as these representations are trivial representations of the $SU(1|m)$ subalgebra.

To see the equivalence in terms of $Osp(2N-2m|2)$, characters, we may, as in section 2, make a change of basis for $Q_{\bar r 1}, S_{\bar s}{}^1, R_{\bar r \bar s}$ to the orthonormal basis, as in (2.8) and (2.9), to $\{\mathcal{Q}_{\bar m 1}, \bar{\mathcal{Q}}_{\bar m 1}, \mathcal{S}_{\bar n}{}^1, \bar{\mathcal{S}}_{\bar n}{}^1, H_{\bar m}, E^{\pm\pm}_{\bar m \bar n}, E^{\pm\mp}_{\bar m \bar n}, \bar m, \bar n = m{+}1, \dots, N\}$, in an obvious way. Denoting highest weight states by $|\bar\Delta; \bar{\mathrm{r}}\rangle^{\text{h.w.}}$, where

$$(K^{11}, S_{\bar s}{}^1)|\bar\Delta; \bar{\mathrm{r}}\rangle^{\text{h.w.}} = 0\,, \qquad (\bar D, H_{\bar m})|\bar\Delta; \bar{\mathrm{r}}\big\rangle^{\text{h.w.}} = (\bar\Delta, r_{\bar m})|\bar\Delta; \bar{\mathrm{r}}\rangle^{\text{h.w.}}\,, \tag{A.23}$$

unitarity requires $\bar\Delta \geq r_{m+1}$. Compatible shortening conditions are given by, for $\bar n = m{+}1, \dots, N$,

$$\begin{aligned} \mathcal{Q}_{\bar n 1}|\bar\Delta; \bar{\mathrm{r}}\rangle^{\text{h.w.}} = 0 \qquad &\Rightarrow \qquad \bar\Delta = r_{m+1} = \ldots = r_{\bar n}\,, \\ \bar{\mathcal{Q}}_{N 1}|\bar\Delta; \bar{\mathrm{r}}\rangle^{\text{h.w.}} = 0 \qquad &\Rightarrow \qquad \bar\Delta = r_{m+1} = \ldots = r_{N-1} = -r_N\,. \end{aligned} \tag{A.24}$$

We may follow a similar procedure as for $Osp(2N|4)$ characters in section 4 to find $Osp(2N|2)$ characters for representations $\mathcal{R}_{(\bar\Delta, \bar{\mathrm{r}})}$. The corresponding characters for irreducible representations are given by,[13]

$$\begin{aligned} &\chi^{(Osp(2N-2m|2),i)}_{(\bar\Delta,\bar{\mathrm{r}})}(\bar s, \bar{\mathrm{y}}) = \mathrm{Tr}_{\mathcal{R}_{(\bar\Delta,\bar{\mathrm{r}})}}(\bar s^{\bar D} \bar y_1{}^{H_{m+1}} \cdots \bar y_{N-m}{}^{H_N}) \\ &= \frac{\bar s^{\bar\Delta}}{1-\bar s^2} \mathfrak{W}^{\mathcal{S}_{N-m} \ltimes (\mathcal{S}_2)^{N-m-1}} \Big(C^{(N-m)}_{\bar{\mathrm{r}}}(\bar{\mathrm{y}}) \mathcal{R}^{(i)}(\bar s, \bar{\mathrm{y}}) \textstyle\prod_{j=1}^{N-m-1}(1 + \bar s\, \bar y_j{}^{-1}) \Big)\,, \end{aligned} \tag{A.25}$$

where, corresponding to the action of supercharges on the highest weight state for long and short representations $\mathcal{R}_{(\bar\Delta,\bar{\mathrm{r}})}$,

$$\mathcal{R}^{(i)}(\bar s, \bar{\mathrm{y}}) = \begin{cases} (1 + \bar s\, (\bar y_{N-m})^{-1}) \prod_{j=1}^{N-m}(1 + \bar s\, \bar y_j)\,, & \text{for } i = \text{long} \\ (1 + \bar s\, (\bar y_{N-m})^{-1}) \prod_{j=\bar n+1-m}^{N-m}(1 + \bar s\, \bar y_j)\,, & \text{for } i = \bar n \\ (1 + \bar s\, (\bar y_{N-m})^{\mp 1})\,, & \text{for } i = \pm, \end{cases} \tag{A.26}$$

where for $i = \text{long}$ then $\bar\Delta \geq r_{m+1}$, for long multiplets, while for $i = \bar n$ then $\bar\Delta = r_{m+1} = \ldots = r_{\bar n} > r_{\bar n+1}$ and for $i = \pm$ then $\bar\Delta = r_{m+1} = \ldots = r_{N-1} = \pm r_n$.

[13] Here the maximal compact subgroup is $U(1) \otimes U(1) \otimes SO(2N{-}2m)$ so that the relevant Weyl symmetriser, acting on Verma module characters, is $\mathfrak{W}^{\mathcal{S}_{N-m} \ltimes (\mathcal{S}_2)^{N-m-1}}$.

Appendix B. Character Expansions

Defining,

$$f(u,x,y) = \sum_{r,s,t\geq 0} N_{rst} u^r \chi_s(x)\chi_t(y) \,, \tag{B.1}$$

and using usual the orthogonality relation for $SU(2)$ characters in (4.5),

$$-\frac{1}{4\pi i}\oint \frac{\mathrm{d}z}{z}(z-z^{-1})^2 \chi_s(z)\chi_t(z) = \delta_{st} \,, \tag{B.2}$$

which may be equivalently expressed by, due to $\chi_j(z) = \chi_j(z^{-1})$,

$$-\frac{1}{2\pi i}\oint \chi_s(z)\chi_t(z)(z-z^{-1})\mathrm{d}z = \delta_{st} \,, \tag{B.3}$$

we have that,

$$N_{rst} = \frac{1}{(2\pi i)^3}\oint\oint\oint f(u,x,y)u^{-r-1}\chi_s(x)\chi_t(y)(x-x^{-1})(y-y^{-1})\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y \,, \tag{B.4}$$

where each contour is the relevant unit circle. Using $\chi_s(z) = -\chi_{-s-2}(z)$, we have,

$$N_{rst} = -N_{r\ -s-2\ t} = -N_{rs\ -t-2} = N_{r\ -s-2\ -t-2} \,. \tag{B.5}$$

It is convenient below to construct generating functions,

$$F_{rs}(z) = \sum_{r=0}^{\infty} N_{r\,r-2s\,r-2t} z^r \,, \tag{B.6}$$

so that using (B.4), summing over r and performing the subsequent contour integral over u for $|z| < |xy|$,

$$\begin{aligned} F_{rs}(z) = \frac{1}{(2\pi i)^2}\oint\oint \Big(& x^{1-2s}y^{1-2t}f(zxy,x,y) - x^{1-2s}y^{-1+2t}f(zx/y,x,y) \\ & - x^{-1+2s}y^{1-2t}f(zy/x,x,y) + x^{-1+2s}y^{-1+2t}f(z/(xy),x,y)\Big)\mathrm{d}x\,\mathrm{d}y \,. \end{aligned} \tag{B.7}$$

Note that for application to the main text, in terms of the notation used below,

$$N_{\mathrm{free},(r,r-s-t,t-s)} = N_{\mathrm{F},r\,r-2s\,r-2t} \,, \qquad N_{\mathrm{sugra},(r,r-s-t,t-s)} = N_{\mathrm{S},r\,r-2s\,r-2t} \,. \tag{B.8}$$

The Free Field Case

For the free field case, due to (7.13) with (7.14), we consider,

$$f_{\mathrm{F}}(u,x,y)=\prod_{j\geq 1}\frac{1}{1-u^j\chi_1(x^j)\chi_1(y^j)}=\sum_{r,s,t\geq 0}N_{\mathrm{F},rst}u^r\chi_s(x)\chi_t(y)\,, \tag{B.9}$$

which may be expanded as,

$$f_{\mathrm{F}}(u,x,y)=\sum_{\underline{\lambda}}u^{|\underline{\lambda}|}p_{\underline{\lambda}}(x,x^{-1})p_{\underline{\lambda}}(y,y^{-1})\,, \tag{B.10}$$

in terms of power symmetric polynomials $p_{\underline{\lambda}}(\mathrm{z})$, (6.9) with (6.4). Using an expansion formula for power symmetric polynomials in terms of Schur polynomials, we may expand the latter in terms of $SU(2)$ characters via,

$$p_{\underline{\lambda}}(z,z^{-1})=\sum_{\substack{m,n\geq 0\\ m+2n=|\underline{\lambda}|}}\omega_{\underline{\lambda}}{}^{(m,n)}\chi_m(z)\,, \tag{B.11}$$

where $\omega_{\underline{\lambda}}{}^{(m,n)}$ are symmetric group characters. Introducing the notation, $\binom{\underline{\lambda}}{\underline{\rho}}=\prod_{j\geq 1}\binom{\lambda_j}{\rho_j}$ then it is easily determined from (B.11) that (other formulae for these characters may be found in in [31] but the following is the most useful for purposes here, and is perhaps simpler),

$$\omega_{\underline{\lambda}}^{(|\underline{\lambda}|,0)}=1\,,\qquad \omega_{\underline{\lambda}}{}^{(|\underline{\lambda}|-2n,n)}=\sum_{\substack{\underline{\rho}\\ |\underline{\rho}|=n}}\binom{\underline{\lambda}}{\underline{\rho}}-\sum_{\substack{\underline{\rho}\\ |\underline{\rho}|=n-1}}\binom{\underline{\lambda}}{\underline{\rho}}\,,\quad n=1,\dots,[|\underline{\lambda}|/2], \tag{B.12}$$

with $\omega_{\underline{\lambda}}^{(n,m)}$ being otherwise zero. Thus, using (B.3), (B.4) for (B.10) with (B.11),

$$N_{\mathrm{F},r\,r-2s\,r-2t}=\sum_{\substack{\underline{\lambda}\\ |\underline{\lambda}|=r}}\omega_{\underline{\lambda}}{}^{(r-2s,s)}\omega_{\underline{\lambda}}{}^{(r-2t,t)}\,,\quad s,t=0,\dots,[r/2]\,. \tag{B.13}$$

Thus, $N_{\mathrm{F},r\,r-2s\,r-2t}$ is a potentially non-zero, non-negative integer only for $r,s=0,\dots,[r/2]$.

A useful identity, which may be easily generalised, is the following, namely, for

$$n_{rij}=\sum_{\substack{\underline{\lambda}\\ |\underline{\lambda}|=r}}\binom{\lambda_j}{i}\,, \tag{B.14}$$

then,[14]

$$g_{ij}(z)=\sum_{r=0}^{\infty}n_{rij}\,z^r=\frac{z^{ij}}{(1-z^j)^i}\prod_{k=1}^{\infty}\frac{1}{1-z^k}\,. \tag{B.15}$$

[14] This may be seen in a simple way by first introducing, $h(x,z)=\frac{1}{1-xz^j}\prod_{\substack{k=1\\k\neq j}}^{\infty}\frac{1}{1-z^k}$, so that, in a series expansion, $h(x,z)=\sum_{\underline{\lambda}}x^{\lambda_j}z^{|\underline{\lambda}|}$. We then have $\lim_{x\to 1}\frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}x^i}h(x,z)=g_{ij}(z)$ using $\frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}x^i}\frac{1}{1-xz^j}=\frac{z^{ij}}{(1-xz^j)^{i+1}}$.

The simplest case is for $s=t=0$, whereby for (B.6) we find, using (B.12) along with (B.15),

$$F_{\mathrm{F},00}(z) = g_{00}(z) = \prod_{k=1}^{\infty} \frac{1}{1-z^k}\,, \tag{B.16}$$

and thus have, giving (7.10),

$$N_{\mathrm{F},rrr} = n_{r00} = p(r)\,. \tag{B.17}$$

Similarly, for (B.6) using (B.12) along with (B.15),

$$F_{\mathrm{F},10}(z) = F_{\mathrm{F},01}(z) = g_{11}(z) - g_{00}(z) = \frac{2z-1}{1-z} \prod_{k=1}^{\infty} \frac{1}{1-z^k}\,, \tag{B.18}$$

and so, agreeing with (7.15),

$$N_{\mathrm{F},r\,r-2\,r} = N_{\mathrm{F},r\,r\,r-2} = n_{r11} - p(r) = \sum_{j=0}^{r-1} p(j) - p(r)\,. \tag{B.19}$$

Similarly, for (B.6) using (B.12) along with (B.15),

$$F_{\mathrm{F},11}(z) = 2g_{21}(z) - g_{11}(z) + g_{00}(z) = \frac{4z^2-3z+1}{(1-z)^2} \prod_{k=1}^{\infty} \frac{1}{1-z^k}\,, \tag{B.20}$$

and so, giving (7.16),

$$N_{\mathrm{F},r\,r-2\,r-2} = 2n_{r21} - n_{r11} + p(r)\,. \tag{B.21}$$

Notice, that by a very similar argument, it may be shown that from (B.12) with (B.13) for $t=0$,

$$\begin{aligned} &F_{\mathrm{F},s\,0}(z) = F_{\mathrm{F},0\,s}(z) \\ &= \sum_{\rho_1,\dots,\rho_s\geq 0} \Big(z^s \delta_{\rho_1+\ldots+s\rho_s,s} - z^{s-1}\delta_{\rho_1+\ldots+s\rho_s,s-1}\Big) \frac{1}{\prod_{j=1}^{s}(1-z^j)^{\rho_j}} \prod_{k=1}^{\infty}\frac{1}{1-z^k}\,, \end{aligned} \tag{B.22}$$

enabling determination of $N_{\mathrm{F},r\,r-2s\,r-2t}$ for $s\,t=0$.

The Supergravity Case

In this case we consider, due to (7.19) with (7.20),

$$f_{\mathrm{S}}(u,x,y) = \prod_{j\geq 1} \frac{1}{\prod_{k,l=0}^{j}(1-u^j x^{2k-j} y^{2l-j})} = \sum_{r,s,t\geq 0} N_{\mathrm{S},rst} u^r \chi_s(x)\chi_t(y)\,, \tag{B.23}$$

and we use (B.7), (B.6), to determine generating functions for low s, t.

For application to (B.6), we have the leading terms,

$$
\begin{aligned}
f_{\mathrm{S}}(zxy, x, y) &= \prod_{k=1}^{\infty} \frac{1}{1-z^k} + \dots\,, \\
f_{\mathrm{S}}(zx/y, x, y) &= \prod_{k=1}^{\infty} \frac{1}{(1-z^k)(1-z^k y^{-2})} + \dots\,, \\
f_{\mathrm{S}}(zy/x, x, y) &= \prod_{k=1}^{\infty} \frac{1}{(1-z^k)(1-z^k x^{-2})} + \dots\,, \\
f_{\mathrm{S}}(z/(xy), xy) &= \prod_{k=1}^{\infty} \frac{1}{(1-z^k)(1-z^k x^{-2})(1-z^k y^{-2})(1-z^k x^{-2} y^{-2})} + \dots\,,
\end{aligned}
\tag{B.24}
$$

where $\dots$ denotes terms that in a series expansion in x, y involve powers x^{2a}, y^{2b}, $a, b \in \mathbb{Z}$, $a, b \neq 0, -1$, that do not contribute below.

We have, from (B.24),

$$
\begin{aligned}
\frac{1}{(2\pi i)^2} \oint \oint xy f_{\mathrm{S}}(zxy, x, y) \mathrm{d}x\, \mathrm{d}y &= 0\,, \\
\frac{1}{(2\pi i)^2} \oint \oint \frac{x}{y} f_{\mathrm{S}}(zx/y, x, y) \mathrm{d}x\, \mathrm{d}y &= 0\,, \\
\frac{1}{(2\pi i)^2} \oint \oint \frac{y}{x} f_{\mathrm{S}}(zy/x, x, y) \mathrm{d}x\, \mathrm{d}y &= 0\,, \\
\frac{1}{(2\pi i)^2} \oint \oint \frac{1}{xy} f_{\mathrm{S}}(z/(xy), x, y) \mathrm{d}x\, \mathrm{d}y &= \prod_{k=1}^{\infty} \frac{1}{1-z^k}\,,
\end{aligned}
\tag{B.25}
$$

so that for (B.6) we have, from (B.7),

$$
F_{\mathrm{S},00}(z) = g_{00}(z)\,. \tag{B.26}
$$

agreeing with (B.16). Hence, $N_{\mathrm{S},rrr} = N_{\mathrm{F},rrr}$ in (B.17), leading to the first equation in (7.21).

Similarly, using (B.24) and

$$
\prod_{k=1}^{\infty} \frac{1}{1-tz^k} = 1 + \frac{zt}{1-z} + O(t^2, z^2)\,, \tag{B.27}
$$

we have,

$$
\begin{aligned}
\frac{1}{(2\pi i)^2}\oint\oint \frac{x}{y} f_{\mathrm{S}}(zxy,x,y)\mathrm{d}x\,\mathrm{d}y &= 0\,,\\
\frac{1}{(2\pi i)^2}\oint\oint yx f_{\mathrm{S}}(zx/y,x,y)\mathrm{d}x\,\mathrm{d}y &= 0\,,\\
\frac{1}{(2\pi i)^2}\oint\oint \frac{1}{xy} f_{\mathrm{S}}(zy/x,x,y)\mathrm{d}x\,\mathrm{d}y &= \prod_{k=1}^{\infty}\frac{1}{1-z^k}\,,\\
\frac{1}{(2\pi i)^2}\oint\oint \frac{y}{x} f_{\mathrm{S}}(z/(xy),x,y)\mathrm{d}x\,\mathrm{d}y &= \frac{z}{1-z}\prod_{k=1}^{\infty}\frac{1}{1-z^k}\,,
\end{aligned}
\tag{B.28}
$$

so that for (B.6), from (B.7),

$$
F_{\mathrm{S},01}(z) = g_{11}(z) - g_{00}(z)\,, \tag{B.29}
$$

agreeing with (B.20), so that $N_{\mathrm{S},r\,r\,r-2} = N_{\mathrm{F},r\,r\,r-2}$ in (B.21). It is easy to show that $F_{\mathrm{S},10}(z) = F_{\mathrm{S},01}(z)$ so that $N_{\mathrm{S},r\,r-2\,r} = N_{\mathrm{S},r\,r\,r-2}$. Thus we have (7.21).

Similarly, using (B.24) with (B.27),

$$
\begin{aligned}
\frac{1}{(2\pi i)^2}\oint\oint \frac{1}{xy} f_{\mathrm{S}}(zxy,x,y)\mathrm{d}x\,\mathrm{d}y &= \prod_{k=1}^{\infty}\frac{1}{1-z^k}\,,\\
\frac{1}{(2\pi i)^2}\oint\oint \frac{y}{x} f_{\mathrm{S}}(zx/y,x,y)\mathrm{d}x\,\mathrm{d}y &= \frac{z}{1-z}\prod_{k=1}^{\infty}\frac{1}{1-z^k}\,,\\
\frac{1}{(2\pi i)^2}\oint\oint \frac{x}{y} f_{\mathrm{S}}(zy/x,x,y)\mathrm{d}x\,\mathrm{d}y &= \frac{z}{1-z}\prod_{k=1}^{\infty}\frac{1}{1-z^k}\,,\\
\frac{1}{(2\pi i)^2}\oint\oint xy f_{\mathrm{S}}(z/(xy),x,y)\mathrm{d}x\,\mathrm{d}y &= \left(\frac{z}{1-z}+\frac{z^2}{(1-z)^2}\right)\prod_{k=1}^{\infty}\frac{1}{1-z^k}\,,
\end{aligned}
\tag{B.30}
$$

so that for (B.6), from (B.7),

$$
F_{\mathrm{S},11}(z) = g_{21}(z) - g_{11}(z) + g_{00}(z) = \frac{3z^2-3z+1}{(1-z)^2}\prod_{k=1}^{\infty}\frac{1}{1-z^k}\,, \tag{B.31}
$$

giving, from (B.15),

$$
N_{\mathrm{S},r\,r-2\,r-2} = n_{r21} - n_{r11} + p(r)\,, \tag{B.32}
$$

leading to (7.22).

Appendix C. Counting Multi-traces of Commuting Matrices

In [26] it was shown that a basis for multiple traces of J commuting $N \times N$ matrices X_k, $k = 1, \ldots, J$, is provided by,

$$\mathrm{Tr}\,\mathcal{U}_1\ \mathrm{Tr}\,\mathcal{U}_2 \cdots \mathrm{Tr}\,\mathcal{U}_N\,, \qquad \mathcal{U}_i = \prod_{j=1}^{n} Y_{ij}\,, \qquad Y_{ij} \in \{X_k,\, k = 0, \ldots, J\}\,, \tag{C.1}$$

for $n \to \infty$, where $X_0 = \mathbb{I}$, the identity matrix, which, of course, also commutes with X_k, $k = 1, \ldots, J$, so that initially we treat it on an equal footing with the other matrices. (C.1) is linearly independent up to the action of the automorphism symmetry group described below.

Each $\mathcal{U}_i$ has an automorphism symmetry $\mathcal{S}_n$ since $\mathcal{U}_i = \prod_{j=1}^{n} Y_{ij} = (\mathcal{U}_i)^\sigma = \prod_{j=1}^{n} Y_{i\sigma(j)}$ for every $\sigma \in \mathcal{S}_n$ (since X_k commute). Hence each $\mathrm{Tr}\,\mathcal{U}_i$ corresponds to a graph with $\mathcal{S}_n$ symmetry K_n where Y_{ij} to corresponds to the j^{th} vertex. Here, K_n is taken to be the complete graph on n vertices.[15] Particular $\mathrm{Tr}\,\mathcal{U}_i$ corresponds to a colouring of the vertices of K_n, in colours c_k, $k = 0, \ldots, J$, with the exact value of $Y_{ij} \in \{X_k,\, k = 0, \ldots, J\}$ corresponding to the colour of the j^{th} vertex.

$\mathrm{Tr}\,\mathcal{U}_1\ \mathrm{Tr}\,\mathcal{U}_2 \cdots \mathrm{Tr}\,\mathcal{U}_N$ corresponds to N copies of K_n, denoted ${K_n}^N$. This graph has a corresponding automorphism group, in graph theory known as the wreath product group, in this case given by the semi-direct product $(\mathcal{S}_n)^N \rtimes \mathcal{S}_N$. For this group, for $\sigma = (\sigma_1, \ldots, \sigma_N) \in (\mathcal{S}_n)^N$, $\tau \in \mathcal{S}_N$, and defining $\sigma^\tau = (\sigma_{\tau^{-1}(1)}, \ldots, \sigma_{\tau^{-1}(N)})$, then for (σ, τ), $(\sigma', \tau') \in (\mathcal{S}_n)^N \rtimes \mathcal{S}_N$, group multiplication is defined by $(\sigma, \tau)(\sigma', \tau') = (\sigma{\sigma'}^{\tau^{-1}}, \tau\tau')$, so that, with $1_{(\mathcal{S}_n)^N \ltimes \mathcal{S}_N} = (1_{\mathcal{S}_n}, \ldots, 1_{\mathcal{S}_n}, 1_{\mathcal{S}_N})$, then the inverse $(\sigma, \tau)^{-1} = ((\sigma^{-1})^\tau, \tau^{-1})$.

The action of $(\mathcal{S}_n)^N \rtimes \mathcal{S}_N$ on (C.1) is given by $(\mathrm{Tr}\,\mathcal{U}_1\ \mathrm{Tr}\,\mathcal{U}_2 \cdots \mathrm{Tr}\,\mathcal{U}_N)^{(\sigma,\tau)} = \mathrm{Tr}(\mathcal{U}_{\tau(1)})^{\sigma_1}\ \mathrm{Tr}(\mathcal{U}_{\tau(2)})^{\sigma_2} \cdots \mathrm{Tr}(\mathcal{U}_{\tau(N)})^{\sigma_N}$, $(\mathcal{U}_i)^{\sigma_k} = \prod_{j=1}^{n} Y_{i\sigma_k(j)}$, with elements of the basis related by such a group transformation being identical. To achieve a linearly independent basis it is necessary to divide out by this automorphism symmetry.

Generically, depending on the graph G, with n vertices, having automorphism group symmetry Γ, the generating function for the number $N_{n_0,\ldots,n_J}$ of colourings of the vertices of G, in n_k colours c_k, $k = 0, \ldots, J$, $\sum_{k=0}^{J} n_k = n$, may be written as,

$$C_\Gamma(x_0, \ldots, x_J) = \sum_{\substack{n_0,\ldots,n_J \geq 0 \\ n_0+\ldots+n_J = n}} N_{n_0,\ldots,n_J}\, {x_0}^{n_0} \cdots {x_J}^{n_J}\,. \tag{C.2}$$

[15] This is the well known graph where any two vertices are joined by an edge. K_n is used here for illustrative purposes - we could in fact consider any graph with $\mathcal{S}_n$ automorphism symmetry, for instance the graph complement of K_n, composed of just n vertices, no edges.

We may use the Polya enumeration theorem to determine the generating function for the number of colourings of ${K_n}^N$, equivalent to the number of independent multiple trace operators, (C.1), subject to the automorphism symmetry $(\mathcal{S}_n)^N \rtimes \mathcal{S}_N$, in terms of cycle indices for the symmetric permutation group.

The cycle index for the symmetric permutation group $\mathcal{S}_n$ is given by, with the notation of section 6,

$$Z_{\mathcal{S}_n}(s_1,\dots,s_n) = \sum_{\substack{\underline{\lambda} \\ |\underline{\lambda}|=n}} \prod_{j=1}^n \frac{1}{j^{\lambda_j}\lambda_j!}{s_j}^{\lambda_j} = \frac{1}{2\pi i}\oint \frac{\mathrm{d}z}{z^{n+1}}\exp\Bigg(\sum_{j\geq 1}\frac{1}{j}z^j s_j\Bigg)\,, \tag{C.3}$$

and the generating function for the number of colourings of K_n is, by the Polya enumeration theorem,

$$\begin{aligned} C_{\mathcal{S}_n}(x_0,x_1,\dots,x_J) &= Z_{\mathcal{S}_n}(s_1,\dots,s_n)\,, \qquad s_i = p_i(x_0,\dots,x_J)\,, \\ &= \frac{1}{2\pi i}\oint \frac{\mathrm{d}z}{z^{n+1}}\frac{1}{(1-zx_0)\dots(1-zx_J)}\,, \end{aligned} \tag{C.4}$$

where $p_i(\mathrm{x}) = \sum_{k=0}^J {x_k}^i$, a power symmetric polynomial. Similarly, for ${K_n}^N$, with automorphism group $(\mathcal{S}_n)^N \rtimes \mathcal{S}_N$, the generating function for the number of colourings is given by, using the Polya enumeration theorem applied to wreath product groups [30],

$$C_{(\mathcal{S}_n)^N\rtimes\mathcal{S}_N}(x_0,x_1,\dots,x_J) = Z_{\mathcal{S}_N}(\widetilde{s}_1,\dots,\widetilde{s}_N)\,, \tag{C.5}$$

where

$$\widetilde{s}_i = Z_{\mathcal{S}_n}(s_{1i},\dots,s_{ni}) = C_{\mathcal{S}_n}({x_0}^i,\dots,{x_J}^i)\,, \qquad s_{ji} = p_j({x_0}^i,{x_1}^i,\dots,{x_J}^i)\,. \tag{C.6}$$

In order that contributions to $\lim_{n\to\infty} C_{\mathcal{S}_n}(x_0,x_1,\dots,x_J)$ from finite numbers of X_k, $k=1,\dots,J$ in $\mathrm{Tr}\,U_i$ not vanish then we must also take $x_0 \to 1$. Following from (C.4),

$$\lim_{n\to\infty} C_{\mathcal{S}_n}(1,x_1,\dots,x_J) = \lim_{s\to 1}(1-s)\sum_{n=0}^\infty s^n C_{\mathcal{S}_n}(1,x_1,\dots,x_J) = \frac{1}{(1-x_1)\cdots(1-x_J)}\,. \tag{C.7}$$

Thus the generating function for the number of distinct basis elements of the form (C.1) as $n\to\infty$ is,

$$\begin{aligned} \lim_{n\to\infty} C_{(\mathcal{S}_n)^N\rtimes\mathcal{S}_N}(1,x_1,\dots,x_J) &= Z_{\mathcal{S}_N}(\widetilde{s}_1,\dots,\widetilde{s}_N)\,, \qquad \widetilde{s}_i = \frac{1}{(1-{x_1}^i)\cdots(1-{x_J}^i)}\,, \\ &= \frac{1}{2\pi i}\oint \frac{\mathrm{d}z}{z^{N+1}}\exp\Bigg(\sum_{n\geq 1}\frac{1}{n}z^n\widetilde{s}_n\Bigg) \\ &= \frac{1}{2\pi i}\oint \frac{\mathrm{d}z}{z^{N+1}}\prod_{j_1,\dots,j_J=0}^\infty \frac{1}{(1-z{x_1}^{j_1}\cdots{x_J}^{j_J})}\,, \end{aligned} \tag{C.8}$$

which matches exactly with the chiral ring partition function obtained using the plethystic approach [25].

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